EXAM 1

Instructions: You will have approximately 50 minutes for this exam. The test is closed book, closed notes and calculators are not allowed. The point value of each problem is written next to the problem – use your time wisely. Partial credit will be given. You may use either pencil or ink. If you need extra paper, raise your hand (I also have a stapler and paper clips for attaching extra sheets). If you have any questions, raise your hand. Please show all work unless instructed otherwise.

Problem 1(15 points) In each of the following cases, you are given a field $\mathbb{F}$, an $\mathbb{F}$-vector space V, and a subset $W \subset V$. State whether or not W is an $\mathbb{F}$-vectors subspace of V. If it is a subspace, compute its dimension. You need not give rigorous justification, but do show all work.

(a)(5 points) $\mathbb{F} = \mathbb{C}$, $V = \mathbb{C}^2$, $W = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} | z_1 + \overline{z}_2 = 0 \right\}$. Here $\overline{z}$ is the complex conjugate, i.e. $\overline{a+ib} = a - ib$.

It is NOT a subspace. To prove this, we will show that W is not stable for scalar multiplication, i.e. we will find a vector v in W and a scalar $\lambda \in \mathbb{C}$ such that λv is not in W.

Consider the vector

$$v = \begin{pmatrix} i \\ i \end{pmatrix}. \tag{1}$$

It is easy to check that v is in W (this is just the identity $i + \overline{i} = i + (-i) = 0$). Let $\lambda = -i$ and consider the scalar multiple

$$\lambda v = -iv = \begin{pmatrix} -i * i \\ -i * i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

It is easy to see that λv is not in W because $1 + \overline{1} = 1 + 1 = 2 \neq 0$. Thus W is not stable for taking scalar multiples.

Subspace?: NO.

Dimension?: N/A.

(b)(5 points) $\mathbb{F} = \mathbb{R}$, $V = \mathbb{C}^2$, $W = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} | z_1 + \overline{z}_2 = 0 \right\}$. Here scalar multiplication of a vector is defined by

$$r \begin{pmatrix} a+ib \\ c+id \end{pmatrix} = \begin{pmatrix} ra+irb \\ rc+ird \end{pmatrix}. \tag{3}$$

In this case W is a vector subspace. First of all observe that the zero vector is in W because of the identity $0 + \overline{0} = 0 + 0 = 0$.

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Next suppose that we have two vectors in W, say

$$v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, w = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}. \tag{4}$$

We will prove that $v + w$ is also in W. Observe that

$$v + w = \begin{pmatrix} z_1 + \zeta_1 \\ z_2 + \zeta_2 \end{pmatrix}. \tag{5}$$

To see that $v + w$ is in W, observe that

$$(z_1 + \zeta_1) + \overline{(z_2 + \zeta_2)} = (z_1 + \zeta_1) + \left(\overline{z_1} + \overline{\zeta_2}\right). \tag{6}$$

This follows since complex conjugation commutes with addition. Because addition is associative, we can rewrite the right-hand side of this equation as

$$(z_1 + \overline{z_2}) + \left(\zeta_1 + \overline{\zeta_2}\right). \tag{7}$$

By the assumption that v and w are in W, each of the terms in parentheses vanishes. So we conclude $v + w$ is also in W.

Finally, suppose that we have a vector v in W (with the same notation as above), and suppose that $r \in \mathbb{R}$ is some real number. We will show that rv is also in W. Indeed, we have

$$rv = \begin{pmatrix} rz_1 \\ rz_2 \end{pmatrix}. \tag{8}$$

And we check that

$$rz_1 + \overline{rz_2} = rz_1 + r\overline{z_2} = r\left(z_1 + \overline{z_2}\right). \tag{9}$$

The first step above follows since scaling by a REAL number commutes with complex conjugation (which is clear from the definition of complex conjugation). The second step follows since multiplication distributes with respect to addition. By the assumption that v is in W, we have that $z_1 + \overline{z_2} = 0$. So we conclude that $r(z_1 + \overline{z_2}) = r0 = 0$, i.e. rv is in W. Thus we have checked the three axioms necessary to prove that W is a vector subspace of V.

Subspace?: YES.

In order to compute the dimension, we write a general vector v in V in the form

$$v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}. \tag{10}$$

Then the condition for v to be contained in W is exactly

$$0 + i0 = (a_1 + ib_1) + \overline{(a_2 + ib_2)} = (a_1 + ib_1) + (a_2 - ib_2) = (a_1 + a_2) + i\left(b_1 - b_2\right). \tag{11}$$

The first step above is by definition of complex conjugation, and the second is by associativity of addition and distributivity of multiplication with addition. This condition is equivalent to the pair of real linear conditions

$$\begin{cases} a_1 + a_2 = 0 \\ b_1 - b_2 = 0 \end{cases} \tag{12}$$

Solving this system of linear equations, we conclude that a basis for W consists of the vectors

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} i \\ i \end{pmatrix}. \tag{13}$$

Therefore W is 2-dimensional.

Dimension?: 2.

(c)(5 points) $\mathbb{F} = \mathbb{R}$, V is the vector space consisting of all 2×2 matrices with real entries

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14}$$

with addition and scalar multiplication defined in the usual way, and $W = \{A \in V : A + A^\dagger = 0\}$. Here $A^\dagger$ denotes the transpose matrix of A.

This is a vector subspace of V. The easiest way to see this is to observe that we may identify V with the vector space $\mathbb{R}^4$, where the basis vectors

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, f_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{15}$$

are identified with the standard basis vectors of $\mathbb{R}^4$, $e_1, \ldots, e_4$ respectively. In other words, we identify a 2×2 matrix with a 4-vector by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{16}$$

Now the condition that $A + A^\dagger = 0$ is precisely that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{17}$$

Gathering terms, this is the same as the four linear conditions

$$\begin{cases} a + a = 0 \\ b + c = 0 \\ c + b = 0 \\ d + d = 0 \end{cases} \tag{18}$$

Identifying matrices with 4-vectors as above, we see that these equations exactly give the nullspace of the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{19}$$

Since the nullspace of a matrix is always a vector subspace, we conclude that W is a vector subspace of V.

Subspace?: Yes.

We can easily put the matrix above in reduced row echelon form by simply scaling the first and third rows by $\frac{1}{2}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

So the rank of this matrix is 3. By the rank-nullity theorem (which simply says that the number of leading variables plus the number of free variables equals the number of total variables), we conclude that the dimension of the nullspace is $4-3=1$. In fact, it is easy to see that a basis for the nullspace is given by the vector/matrix

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21}$$

Dimension?: 1.

Problem 2(20 points) In this problem, $\mathbb{F} = \mathbb{R}$. Consider the real vector space $V = \mathbb{R}^5$. Define the vector subspaces $W_1 \subset V$ and $W_2 \subset V$ as follows. We define W_1 to be the span of the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{22}$$

We define W_2 to be the span of the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{23}$$

Using any method you like (as long as you show your work), compute a spanning set for the vector subspace $W_1 \cap W_2 \subset V$.

Of course we could use our algorithm from the homework to find the intersection. But it is much simpler to observe that the vectors in W_1 are simply those of the form

$$\begin{pmatrix} X_1 \\ 0 \\ X_2 \\ 0 \\ X_3 \end{pmatrix}. \tag{24}$$

On the other hand, every vector in W_2 is a linear combination

$$Y_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + Y_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + Y_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} Y_1 + Y_2 \\ -Y_1 + Y_2 \\ Y_1 + Y_2 + Y_3 \\ -Y_1 + Y_3 \\ Y_1 + Y_3 \end{pmatrix}. \tag{25}$$

The condition that this vector be contained in W_1 is exactly the vanishing of the two coordinates:

$$-Y_1 + Y_2 = 0, -Y_1 + Y_3 = 0. \tag{26}$$

We may rewrite these two conditions as $Y_2 = Y_1$ and $Y_3 = Y_1$. Substituting Y_1 in for Y_2 and Y_3, we conclude that the vectors in $W_1 \cap W_2$ are precisely those of the form

$$\begin{pmatrix} 2Y_1 \\ 0 \\ 3Y_1 \\ 0 \\ 2Y_1 \end{pmatrix}. \tag{27}$$

So we conclude that a basis for $W_1 \cap W_2$ consists of the single vector

$$\begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}. \tag{28}$$

Problem 3(20 points) In each case below, find all solutions to the system of equations $AX = C$. Please show all work, but you don't need to give a rigorous proof of your answer.

(a)(10 points)

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}. \tag{29}$$

We form the augmented matrix A':

$$A' = \left(\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 1 & 1 & 2 & 9 \end{array}\right). \tag{30}$$

Now we perform Gauss-Jordan elimination. We begin by setting $B_1 = A'$. First we transpose the first and second row to get a leading 1 in the first column:

$$B_2 = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 2 & 1 & 9 \end{array}\right). \tag{31}$$

Next we subtract 2 times the first row from the second row to get

$$B_3 = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & -3 & -9 \end{array}\right). \tag{32}$$

Next we scale the second row by $\frac{-1}{3}$ to get the row echelon matrix

$$B_4 = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right). \tag{33}$$

Finally we subtract 2 times the second row from the first row to get a reduced row echelon matrix

$$R = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right). \tag{34}$$

Written using variables instead of matrix notation, the reduced row echelon system is

$$R : \left\{ \begin{array}{l} 1x_1 + 1x_2 + 0x_3 = 3 \\ 0x_1 + 0x_2 + 1x_3 = 3 \end{array} \right. \tag{35}$$

Here the leading variables are x_1 and x_3 and the free variable is x_2. We bring the terms involving free variables to the right-hand side of the equation to get

$$\begin{cases} x_1 = 3 - x_2 \\ \qquad x_3 = 3 \end{cases} \tag{36}$$

Therefore the general solution of the system is

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 3 - x_2 \\ x_2 \\ 3 \end{pmatrix}. \tag{37}$$

(b)(10 points)

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}. \tag{38}$$

We could perform Gauss-Jordan elimination to this matrix to solve the problem, but actually there is no need. The first row of the matrix gives the equation

$$2X_1 + X_2 = 2. \tag{39}$$

The second row of the matrix gives the equation

$$2X_1 + X_2 = 4. \tag{40}$$

But the left-hand side of each equation is the same. So, if there were any solutions we would have, by transitivity of equality, that $2 = 4$, which is absurd. Therefore we conclude that there are no solutions to this system of equations.

Problem 4(15 points) Over the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, the two-element field, consider the vector space $V = \mathbb{F}_2^3$. Define the two vectors $v_1, v_2 \in V$ by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{41}$$

Write down all vectors $v \in V$,

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{42}$$

such that v_1, v_2, v forms a basis for V. Please show all work and explain your answer, but you need not give a 100% rigorous proof.

(Hint: There are exactly four such vectors.)

It is actually simpler (conceptually anyway) to try to find the vectors w such that v_1, v_2, w DOESN'T form a basis. The complement of this set in V will be the set of vectors we are looking for.

Since the dimension of V is 3, a theorem from the book tells us that any set of 3 linearly independent vectors is a basis for V. So v_1, v_2, w fails to be a basis precisely when v_1, v_2, w are linearly dependent. On the other hand, clearly v_1, v_2 are linearly independent (this is "visible" by observing that each has a zero coordinate where the other has a coordinate equal to one). By another theorem from the book, adding a new vector to a linearly independent set of vectors gives a linearly dependent set if and only if the new vector is in the span of the other vectors. So v_1, v_2, w is linearly dependent iff w is in the span of v_1, v_2, i.e.

$$w = c_1v_1 + c_2v_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_1 \end{pmatrix}, \tag{43}$$

for some pair of scalars $c_1, c_2 \in \mathbb{F}_2$. Another way to characterize this set is as the collection of vectors whose first coordinate is equal to the third coordinate. So the complement of this set is the collection of vectors whose first coordinate is different from the third coordinate. So the set of vectors v for which v_1, v_2, v is a basis for V is the following set:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{44}$$

Problem 5(30 points) **(a)**(10 points) Given an $m \times n$ matrix A over a field $\mathbb{F}$, define the *nullspace* of A.

The nullspace of A is defined to be the set of vectors $X \in \mathbb{F}^n$ such that $AX = 0$.

In each of the following parts, state whether the sentence is true or false. If the sentence is false, provide a counterexample (and say a word or two about why the counterexample works). If the sentence is true, explain why it is true. Your arguments should be convincing, but need not be 100% rigorous.

(b)(10 points) If U is an invertible $n \times n$ matrix, then the nullspace of A equals the nullspace of AU.

This is false. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{45}$$

The nullspace has basis

$$X = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{46}$$

On the other hand, consider the invertible matrix (which is even an elementary matrix):

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{47}$$

The product of A and U is the matrix

$$AU = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{48}$$

The nullspace of AU has basis

$$Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{49}$$

Since Y is not contained in the span of X, we conclude that the nullspace of A does not equal the nullspace of AU.

(c)(10 points) If U is an invertible $m \times m$ matrix, then the nullspace of A equals the nullspace of UA.

This is true. For one thing, this is a theorem in the text (which you are allowed to quote without citing the specific number). If we wanted, this is also quite simple to prove: we will prove that the nullspace of UA contains the nullspace of A and vice versa.

To see that the nullspace of UA contains the nullspace of A, observe that for any vector X in the nullspace of A, we have

$$(UA)X = U(AX) = U0 = 0. \tag{50}$$

The first step follows from associativity of matrix multiplication, and the second step follows from the assumption that X is in the nullspace of A. We conclude that X is in the nullspace of UA. Therfore the nullspace of UA contains the nullspace of A. The same argument works in reverse by observing that $A = U^{-1}(UA)$, so A is the product of UA with an invertible matrix on the left. We conclude that the nullspace of A equals the nullspace of UA.

Extra Credit(5 points) Only attempt this if you have solved all of the other problems and checked your answers. Prove that for an $m \times n$ matrix A and an $n \times p$ matrix B, the rank of AB is at most the minimum of the rank of A and the rank of B. Please justify your answer, but your argument need not be 100% rigorous.

A theorem proved in lecture is that, with the notation from above, the rowspace of AB is contained in the rowspace of B. Therefore the dimension of the rowspace of AB is at most the dimension of the rowspace of B, i.e. $\text{rank}(AB) \leq \text{rank}(B)$. It is also a theorem from the book that the rank of a matrix is the same as the rank of its transpose, in particular $\text{rank}(AB) = \text{rank}(AB)^\dagger$. Another theorem from the book (just a simple calculation really), shows that the transpose of a product of matrices equals the product of the transposes in the opposite order, in particular $(AB)^\dagger = B^\dagger A^\dagger$. Now again using the first theorem mentioned above, we conclude that $\text{rank}(B^\dagger A^\dagger) \leq \text{rank}(A^\dagger)$. And once again using the second theorem mentioned above, we know that $\text{rank}(A^\dagger) = \text{rank}(A)$. Putting the pieces together, we have

$$\text{rank}(AB) = \text{rank}(AB)^\dagger = \text{rank}(B^\dagger A^\dagger) \leq \text{rank}(A^\dagger) = \text{rank}(A). \tag{51}$$

So we have both inequalities $\text{rank}(AB) \leq \text{rank}(B)$ and also $\text{rank}(AB) \leq \text{rank}(A)$. So the rank of AB is at most the minimum of the rank of A and the rank of B.

Have a great weekend, see you next week!