Notes on Lectures 9-10 (September 30, 2005)

Let V and W be two $\mathbb{F}$-vector spaces. Then there is a natural structure of vector space on $V \times W = \{(v, w) : v \in V, w \in W\}$, as follows

$$\begin{aligned}(v_1, w_1) + (v_2, w_2) &:= (v_1 + v_2, w_1 + w_2)\\ k(v_1, w_1) &:= (kv_1, kw_1),\end{aligned} \tag{0.0.1}$$

for any $v_1, v_2 \in V$, $w_1, w_2 \in W$, $k \in \mathbb{F}$. It is easy to check that the axioms in the definition of a vector space (definition 1.2.4.) hold.

V can be viewed as the subspace $\{(v, 0) : v \in V\}$ of $V \times W$ and W as the subspace $\{(0, w) : w \in W\}$.

Assume V is n-dimensional, with a basis $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ and W is m-dimensional, with a basis $\mathcal{B}_W = \{w_1, \ldots, w_m\}$. Then a basis for $V \times W$ is given by $\mathcal{B}_{V\times W} = \{(v_i, 0) : 1 \le i \le n\} \cup \{(0, w_j) : 1 \le j \le m\}$, which implies that the dimension of $V \times W$ is $m + n$. Let us check that $\mathcal{B}_{V\times W}$ is indeed a basis.

linear independence: Consider the linear relation $\sum_{i=1}^n a_i(v_i, 0)+\sum_{j=1}^m b_j(0, w_j) = (0, 0)$. This implies $(\sum_{i=1}^n a_iv_i, \sum_{j=1}^m b_jw_j) = (0, 0)$ and therefore $\sum_{i=1}^n a_iv_i = 0$ and $\sum_{j=1}^m b_jw_j = 0$. Since $\{v_1, \ldots, v_n\}$ in V, respectively $\{w_1, \ldots, w_m\}$ in W are linearly independent, it follows that $a_i = 0$, for all $1 \le i \le n$, respectively $b_j = 0$, for all $1 \le j \le m$. This proves the linear independence.

spanning set: Let (v, w) be an element in $V \times W$. Then $v \in V$ can be written as a linear combination $v = \sum_{i=1}^n v_i$ because $\{v_1, \ldots, v_n\}$ span V. Similarly w can be written as $w = \sum_{j=1}^m w_j$. Then $(v, w) = \sum_{i=1}^n a_i(v_i, 0) + \sum_{j=1}^m b_j(0, w_j)$. This implies that $\mathcal{B}_{V\times W}$ is a spanning set for $V \times W$.

Also remark that if V is a subset of some $\mathbb{F}^k$ and W is a subspace of an $\mathbb{F}^p$, then $V \times W$ is a subspace of $\mathbb{F}^{k+p}$.

A homework exercise asked to show that if V and W are subspaces of the same vector space U, then $V + W = \{v + w : v \in V, w \in W\}$ and $V \cap W$ are also subspaces of the same vector space U. Note that $V \cup W$ is not a subspace in general. For example, if V and W are two lines containing the origin of $\mathbb{R}^3$ (so they are subspaces), $V \cup W$ is the union of the two lines, so it is not a subspace (there are vectors $v \in V$ and $w \in W$ such that $v + w \notin V \cup W$). In this example $V + W$ is the plane containing the two lines, and $V \cap W$ is just a point, the origin.

Let us show that if V and W are finite dimensional (of dimensions n, respectively m) than

$$dim(V) + dim(W) = dim(V \cap W) + dim(V + W).$$

In the example above, this is $1 + 1 = 2 + 0$.

Since $V \cap W$ is a subspace of V (or W), it is finite dimensional. Assume its dimension is $p \le min\{n, m\}$ and let $\{v_1, \ldots, v_p\}$ be a basis for $V \cap W$. The set $\{v_1, \ldots, v_p\}$ is linearly independent as a subset of V, so it can be extended to a basis of V, $\{v_1, \ldots, v_p, v_{p+1}, \ldots v_n\}$. Similarly there is a basis of W, $\{v_1, \dots, v_p, w_{p+1}, \dots, w_m\}$. The claim would follow if we showed that the set $\{v_1, \dots, v_p, v_{p+1}, \dots, v_n, w_{p+1}, \dots, w_m\}$ were a basis for $V + W$.

linearly independent: consider the linear relation $a_1v_1 + \cdots + a_pv_p + \dots a_nv_n + b_{p+1}w_{p+1} + \dots b_mw_m = 0$. This implies that $a_1v_1 + \dots a_nv_n = -(b_{p+1}w_{p+1} + \dots b_mw_m)$. Call this element x. The left hand side implies that $x \in V$ and the right hand side that $x \in W$. So $x \in V \cap W$ and therefore x can be expressed as a linear combination of the basis vectors of $V \cap W$: $x = c_1v_1 + \cdots + c_pv_p$. Then $a_1v_1 + \cdots + a_pv_p + a_{p+1}v_{p+1} + \dots a_nv_n = c_1v_1 + \cdots + c_pv_p$, and so $(a_1 - c_1)v_1 + \cdots + (a_p - c_p)v_p + a_{p+1}c_{p+1} + \dots a_nc_n = 0$. Since $\{v_1, \dots, v_n\}$ is a basis for V, and in particular linearly independent, it follows that $a_1 = c_1, \dots, a_p = c_p$ and $a_{p+1} = \cdots = a_n = 0$.

Going back to the relation we started with and using $a_{p+1} = \cdots = a_n = 0$, it implies that $a_1v_1 + \cdots + a_pv_p + b_{p+1}w_{p+1} + \dots b_mw_m = 0$. Since $\{v_1, \dots, v_p, w_{p+1}, \dots, w_m\}$ is a basis of W, it follows that $a_1 = \cdots = a_p = 0$ and $b_{p+1} = \cdots = b_m = 0$. In conclusion all scalars in the linear relation must be zero, and therefore $\{v_1, \dots, v_p, v_{p+1}, \dots, v_n, w_{p+1}, \dots, w_m\}$ is linearly independent.

spanning set: exercise.