<u>Notes on Lecture 6</u> (September 21, 2005)

This is a slightly different proof of the fact that an invertible matrix is necessarily square without using explicitly the rank. Let A be an $m \times n$ matrix. Recall that A is *invertible* if there exists a matrix B such that $BA = I_n$ and $AB = I_m$. We showed in class that, if it exists, the inverse is unique.

The *transpose* of A is the matrix A^t defined as follows $A^t(i,j) = A(j,i)$. If A is $m \times n$, then A^t is $n \times m$. For example, if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then $A^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Exercise 1. A is invertible if and only if A^t is invertible.

Solution. Using the fact that $(AB)^t = B^t A^t$, show that B is the inverse of A if and only if B^t is the inverse of A^t.

Exercise 2. If A is invertible, then A is a square matrix.

Solution. Assume A is $m \times n$. Consider the linear system $AX = b$. Since A is invertible, we know the system has a unique solution. One of the exercises in PSet 1 asked to show that if $m < n$, and the system is consistent, then it has infinitely many solutions. This implies that in our case, $m \geq n$.

Now, apply the same argument for A^t. A^t is invertible, so the system $A^t X = b$ has a unique solution. But A^t is $n \times m$, and so $n \geq m$.

Therefore $m = n$.

The result proven in class and this fact imply the following characterization.

Proposition. *A matrix is invertible if and only if it is square and it has full rank.*