Inequalities: Hölder's and Minkowski's. November 30, 2005

Let us fix $V = \mathbb{R}^n$ and the usual inner product (dor product) on $\mathbb{R}^n$: if $u = (a_1, \dots, a_n)$ and $v = (b_1, \dots, b_n)$, then $\langle u, v\rangle = a_1b_1 + \cdots + a_nb_n$.

Recall the definition of the *p-norm* ($p \geq 1$): $||u||_p = (\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}}$. If $p = 2$, we recover the usual length function in $\mathbb{R}^n$, and for $p = 1$, we get the *taxicab* norm.

Hölder's Inequality. Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^n |a_ib_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^q\right)^{\frac{1}{q}}.$$

Proof. Using the notation above $u = (a_1, \dots, a_n)$, $||u||_p = (\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}}$, and similarly for v, note that the right hand side of the inequality is $||u||_p \cdot ||v||_q$. If $||u||_p = 0$, then all $a_i = 0$ and the inequality holds trivially. Similarly if $||v||_q = 0$. So let us assume that $||u||_p > 0$ and $||v||_q > 0$.

If we denote by $u' = \frac{1}{||u||_p}u = (a'_1, \dots, a'_n)$ and $v' = \frac{1}{||v||_q}v = (b_1, \dots, b'_n)$, then dividing both sides of the inequality by $||u||_p \cdot ||v||_q$, we see that it is equivalent to proving that

$$\sum_{i=1}^n |a'_ib'_i| \leq 1.$$

Note that $||u'||_p = \frac{1}{||u||_p}||u||_p = 1$, and similarly $||v'||_q = 1$.

If $|a'_i| > 0$, then it can be written as $|a'_i| = e^{\frac{s}{p}}$, for some $s \in \mathbb{R}$. Similarly write $|b'_i| = e^{\frac{t}{q}}$, for some $t \in \mathbb{R}$. Note that $|a'_i|^p = e^s$ and $|b'_i|^q = e^t$. Then

$$|a'_i||b_i|' = e^{\frac{s}{p}+\frac{t}{q}} \leq \frac{1}{p}e^s + \frac{1}{q}e^t = \frac{1}{p}|a'_i|^p + \frac{1}{q}|b'_i|^q.$$

Here, we used the fact that the exponential is a convex function, i.e., $e^{ax+(1-a)y} \leq ae^x + (1-a)e^y$, for all $x, y \in \mathbb{R}$ and that $\frac{1}{q} = 1 - \frac{1}{p}$.

Summing, we find that $\sum_{i=1}^n |a'_i||b'_i| \leq \frac{1}{p}\sum_{i=1}^n |a'_i|^p + \frac{1}{q}\sum_{i=1}^n |b'_i|^q = \frac{1}{p}||u'||_p^p + \frac{1}{q}||v'||_q^q = \frac{1}{p} + \frac{1}{q} = 1$.

□

Minkowski's Inequality. If $p \geq 1$,

$$\left(\sum_{i=1}^n |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |b_i|^p\right)^{\frac{1}{p}}.$$

Proof. If $p = 1$, it follows immediately from the triangle inequality for the absolute value. Assume $p > 1$. Then $\sum_{i=1}^n |a_i + b_i|^p = \sum_{i=1}^n |a_i + b_i| \cdot |a_i + b_i|^{p-1} \leq \sum_{i=1}^n |a_i||a_i + b_i|^{p-1} + \sum_{i=1}^n |b_i||a_i + b_i|^{p-1}$, by using the triangle inequality for the absolute value.

Set $q = \frac{p}{p-1}$, so that $\frac{1}{p} + \frac{1}{q} = 1$. Applying Hölder inequality, we get

$$\sum_{i=1}^n |a_i||a_i + b_i|^{p-1} \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |a_i + b_i|^{(p-1)q}\right)^{\frac{1}{q}},$$

and similarly for b_i's.

It follows that $\sum_{i=1}^n |a_i+b_i|^p \leq \left((\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^n |b_i|^p)^{\frac{1}{p}} \right) \left(\sum_{i=1}^n |a_i + b_i|^{(p-1)q}\right)^{\frac{1}{q}} = \left((\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^n |b_i|^p)^{\frac{1}{p}} \right) (\sum_{i=1}^n |a_i + b_i|^p)^{\frac{p-1}{p}}$. Dividing by $(\sum_{i=1}^n |a_i + b_i|^p)^{\frac{p-1}{p}}$, the claim follows.

$\square$

Note that the Minkowski's inequality can be regarded as the triangle inequality for the p-norm.