<u>**Notes on Lecture 3**</u> **(September 12, 2005)**

1. In class, I mentioned the following exercise:

Proposition. *Fix the positive integer m. If M is an $n \times n$ matrix with the property that*

$$A \cdot M = A, \quad \text{for all } m \times n \text{ matrices } A,$$

then $M = I_n$ (the identity matrix).

Proof. $A \cdot M = A$ for all A. The idea is to choose some particular A. Let $A = E_{1q}$, where E_{1q} is the $m \times n$ matrix whose entries are all zero, except the $(1, q)$-entry, which is 1. Check that, if $M = (m_{ij})$, then $E_{1,q}M$ is the matrix whose rows are all zero, except the first row, which equals the q-th row of M. For example, for 3×3 matrices,

$$E_{12}M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} m_{21} & m_{22} & m_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since we require that $E_{1q}M = E_{1q}$, it follows that $m_{ql} = 0$ for all $l \neq q$, and $m_{qq} = 1$. Varying q, $1 \leq q \leq n$, we find that all off diagonal entries of M have to be zero and all diagonal entries are 1. This implies $M = I_n$.

□

2. In the proof of associativity for matrix multiplication, the key step are the equalities:

$$\sum_{k=1}^{p} \sum_{j=1}^{n} A(i,j)B(j,k)C(k,l) = \sum_{j=1}^{n} \sum_{k=1}^{p} A(i,j)B(j,k)C(k,l) =$$

$$= \sum_{j=1}^{n} A(i,j) \sum_{k=1}^{p} B(j,k)C(k,l).$$

For the second equality, note that $A(i,j)$ does not depend on k, so it can be factored outside the sum $\sum_{k=1}^{p}$.

The first equality comes from changing the order of summation. This is correct, since we have finite sums. For infinite sums (a.k.a. series), one can change the order of summation only in series which converge absolutely. For example, one can sum the terms of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

in any order, and the result would still be $\frac{\pi^2}{6}$.

But for the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots,$$

which converges to $\ln 2$ (but it is not absolutely convergent), changing the order of the summation will affect the result (in fact, if we pick any constant

c, it is possible to change the order in which we sum terms in this series such that the resulting sum is c).