Permutations. November 5, 2005

The purpose of this note is to summarize some properties of the permutations. Recall that a *permutation of* n is a string $\sigma = (\sigma_1, \dots, \sigma_n)$, where $\sigma_i \in \{1, 2, \dots, n\}$ and $\sigma_i \neq \sigma_j$ is $i \neq j$. A better definition is to say that a permutation σ is a bijective function

$$\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}, \quad \sigma(i) = \sigma_i.$$

Denote the set of all permutations of n by S_n. A distinguished element of S_n is the *identity permutation*, $\sigma = (1, 2, \dots, n)$.

If $\sigma = (\sigma_1, \dots, \sigma_n) \in S_n$, we say that (σ_i, σ_j) is an *inversion* if $\sigma_i > \sigma_j$ and $i < j$. Denote by $\ell(\sigma)$ the number of inversions in the permutation σ. Then the *sign* of σ is $sg(\sigma) = (-1)^{\ell(\sigma)}$, and we say that σ is an *even* (*odd*) permutation if $sg(\sigma) = 1$ (respectively, -1).

Example. In S_6, consider $\sigma = (5, 6, 4, 2, 3, 1)$. Then $\ell(\sigma) = 10$, and $sg(\sigma) = 1$.

We record some properties of permutations.

1. It is possible to define *multiplication* of two permutations. This is clear if we think of permutations as functions; then the multiplication is the composition of functions. In the other notation, if $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$ are two permutation, define the product $\sigma \cdot \tau$ to be the permutation $\sigma \cdot \tau = (\sigma_{\tau_1}, \sigma_{\tau_2}, \dots, \sigma_{\tau_n})$. For example, if $\sigma = (4, 2, 3, 1)$ and $\tau = (3, 1, 2, 4)$, then $\sigma \cdot \tau = (3, 4, 2, 1)$, and $\tau \cdot \sigma = (4, 1, 2, 3)$. Note that multiplication of permutations is not commutative, but it is associative (because the composition of functions is).

2. Every permutation has an inverse. If $\sigma \in S_n$ is a permutation, then there exists a permutation $\sigma^{-1} \in S_n$ such that $\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = 1_n$, where by 1_n we denoted here the identity permutation in S_n. Again, this statement is clear if we think of permutations as bijective functions. For example, in S_4, the inverse of the permutation $\sigma = (4, 2, 3, 1)$ is $\sigma^{-1} = (4, 2, 3, 1)$ (so $\sigma \cdot \sigma = 1_4$) and the inverse of $\tau = (3, 1, 2, 4)$ is $\tau^{-1} = (2, 3, 1, 4)$. In general, the inverse of a permutation σ has i on the σ_i position.

3. From 1 and 2, we see that the set of permutations S_n has an identity element, a multiplication of elements and every element has an inverse. In general, a set G, together with an operation $\star$, which satisfies the axioms:

(1) $\star$ is associative;
(2) there exists a *unit* element $e \in G$, such that $x \star e = e \star x = x$, for all $x \in G$.
(3) if $x \in G$, there exists $x^{-1} \in G$ (the inverse) such that $x \star x^{-1} = x^{-1} \star x = e$.

is called a *group*. So S_n is the *group of permutations of* n (or the *symmetric group*).

4. One homework exercise asked you to identify a connection between the permutation of n and $n \times n$ permutation matrices, i.e. matrices which have a 1 on each row and column and zeros everywhere else. If $\sigma = (\sigma_1, \dots, \sigma_n)$ is a permutation of n, define the permutation matrix P_σ to be the matrix such that

$$P_\sigma(\sigma_j, j) = 1 \quad \text{and } P_\sigma(i, j) = 0, \text{ if } j \neq \sigma_i.$$

That is, in the column j, we put one on the σ_j position, and zeros elsewhere. Note that, a priori, one could define the permutation matrix by applying the procedure to rows instead of columns. But that definition will not behave well with respect to multiplication.

Verify that $sg(\sigma) = det(P_\sigma)$ and $P_{\sigma\cdot\tau} = P_\sigma \cdot P_\tau$. From this, it also follows that $sg(\sigma\cdot\tau) = sg(\sigma)sg(\tau)$ and that $sg(\sigma^{-1}) = sg(\sigma)$. In particular, the product of two even permutations is an even permutation, and the inverse of an even permutation is also even. So the even permutations form a *subgroup* of S_n (note that the odd permutations don't).

5. An important class of permutations is the *transpositions*. A permutation σ is called said to be a transposition (ij), if $\sigma_i = j$, $\sigma_j = i$ and $\sigma_k = k$, for all $k \neq i, j$. Note that each transposition is its own inverse. Their importance comes from the fact that every permutation can be written as a product of transpositions. This fact is not hard to prove by induction. The idea is that, given a permutation σ of n, the number n appears in some position in σ, say $\sigma_j = n$. Then by multiplying σ by transpositions we can move n to the nth position and regard the resulting permutation as a permutation of $n-1$. Then apply the induction. It would be a good exercise to try to write down this idea into a formal proof.

For example $\sigma = (4, 2, 3, 1) = (14)$ and $\tau = (3, 1, 2, 4) = (23) \cdot (12)$.

There are many other beautiful and important properties of permutations (for example, the cycle decomposition of a permutation), but we will leave them for some other time.