# 18.700. Final Exam. Fall 2005.

Name: ______________________ December 22, 2005

Problem 1: _____/30

Problem 2: _____/40

Problem 3: _____/20

Problem 4: _____/40

Problem 5: _____/25

Problem 6: _____/20

Problem 7: _____/30

Problem 8: _____/30

Problem 9: _____/35

Problem 10:_____/30

**Total**: _____ /300

**Instructions:** The exam is closed book, closed notes and calculators are not allowed. You will have 170 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

Good luck!

**Problem 1 (30 points)** Determine if the following systems are consistent. If so, determine all solutions.

a) (15 pts)

$$\begin{cases} -X & & & + & 2Z & = & 1 \\ 3X & - & Y & + & 3Z & = & 5 \\ 2X & + & 3Y & - & Z & = & -2 \end{cases}$$

b) (15 pts)

$$\begin{cases} 2X & & + & 5Z & - & W & = & 9 \\ X & - & Y & + & Z & + & 2W & = & 1 \end{cases}$$

**Problem 2 (40 pts)** Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 4 & 2 & -1 \\ 3 & 1 & 5 & 6 & 0 \\ 1 & 1 & 1 & 2 & -1 \end{pmatrix}.$$

a) (20 pts) Find the reduced row echelon form of $A$.

b) (10 pts) Find bases for the column space and for the row space of $A$.

c) (10 pts) Find a basis for the orthogonal complement (in $\mathbb{R}^5$) of the row space of $A$.

**Problem 3 (20 points)** The following matrix $A$ has rank 3. Find a *right* inverse for $A$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 3 & -1 & 0 & 0 & -1 \\ 3 & -1 & 3 & 1 & -1 \end{pmatrix}.$$

**Problem 4 (40 points)** Let $P_4$ be the vector space of polynomials in $X$ of degree at most 4 with real coefficients. Consider the following vectors in $P_4$:

$v_1 = X^4$, $v_2 = (X-1)^2X(X+1)$, $v_3 = (X-1)X^2(X+1)$, $v_4 = (X-1)X(X+1)^2$.

Set $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$.

a)(20 pts) Find a subset $\mathcal{C} \subset \mathcal{B}$ that is a basis for $span(\mathcal{B})$.

b) (10 pts) Express every element of $\mathcal{B}$ in coordinates with respect to $\mathcal{C}$.

c) (10 pts) Extend the set $\mathcal{C}$ to a basis of $P_4$.

**Problem 5 (25 points)** Consider the matrix:

$$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -3 \\ -2 & -3 & -3 & 9 \end{pmatrix}.$$

Find the determinant of $A$ (say which method you are using).

**Problem 6 (20 points)** Let $A$ be a $5 \times 5$ matrix such that all entries of $A$ are even integers.

a)(10 points) Can the determinant of $A$ be 120? (Explain)

b) (10 points) Assume the determinant of $A$ is 160. If the entries of $A^{-1}$ are written as fractions in lowest terms, what is the maximum possible value of the denominators? (Explain)

**Problem 7 (30 points)** Let $V$ be the $\mathbb{R}$-vector space of differentiable real-valued functions on $\mathbb{R}$ with the ordered basis $\mathcal{B} = (v_1, v_2, v_3)$, where

$$v_1 = (e^x + e^{-x})\cos(x), \ v_2 = \cos(x), \ v_3 = (e^x - e^{-x})\sin(x).$$

Let $W$ be the $\mathbb{R}$-vector space of differentiable real-valued functions on $\mathbb{R}$ with the ordered basis $\mathcal{C} = (w_1, w_2, w_3)$, where

$$w_1 = (e^x - e^{-x})\cos(x), \ w_2 = \sin(x), \ w_3 = (e^x + e^{-x})\sin(x).$$

Let $T : V \to W$ be the linear transformation $T(f(x)) = f'(x)$. Compute the matrix representative of $T$ with respect to the ordered bases $\mathcal{B}$ for $V$ and $\mathcal{C}$ for $W$.

**Problem 8 (30 points)** For the following (complex) matrix $A$, compute the Jordan block form $J$ of $A$, and find a transition matrix $P$, such that $A = PJP^{-1}$.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

**Problem 9 (35 points)** Consider the real symmetric matrix

$$S = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Determine a diagonal matrix $D$ and an orthogonal matrix $Q$ such that $S = QDQ^t$.

**Problem 10 (30 points)**

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

Find the $QR$ decomposition of $A$.