Linear operators of $\mathbb{R}^2$. November 7, 2005

The purpose of this note is to illustrate the notion of *linear operators*, by looking at some examples of linear transformations of the plane, $T : \mathbb{R}^2 \to \mathbb{R}^2$. We will consider the standard basis of $\mathbb{R}^2$ and all matrices associated to a linear transformation will be written with respect to this basis.

1. **Rotations:** Let R_θ be the linear transformation given by rotation counterclockwise by the angle θ. The matrix associated to R_θ is

$$[R_\theta] = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

This is clearly an invertible transformation and the inverse is the rotation by θ clockwise, or $R_{-\theta}$. The composition of two rotations $R_{\theta_1} \circ R_{\theta_2}$ is again a rotation $R_{\theta_1+\theta_2}$. Note that the composition of rotations is commutative and that the rotations form a group (the unit element is the identity operator, or the rotation by angle 0). This group is called the *special orthogonal group* of $\mathbb{R}^2$, and it is denoted by $SO(2)$.

2. **Dilations:** For $a, b > 0$, let $D_{a,b}$ denote the linear transformation which takes a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ into the vector $\begin{pmatrix} ax \\ by \end{pmatrix}$. The matrix associated is diagonal, $[D_{a,b}] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Since $a \neq 0$ and $b \neq 0$, $D_{a,b}$ is invertible and the inverse is the dilation $D_{a^{-1},b^{-1}}$.

3. **Shear transformations:** typical examples are the shear transformations parallel with the x-axis, that is given by matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Such a transformation moves the tip of a vector v parallel with the x-axis (in general parallel to a line) and fixes the vectors in the x-axis. They are invertible transformations and the inverses are given by the opposite shear transformation, e.g. in this case $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$.

4. **Projections:** having fixed a line L, the map P_L defined by projecting a vector v on the line L is a linear transformation. Denote the projection on the x-axis by P_x and similarly, the projection onto the y-axis by P_y. The projections are not invertible. For example, $[P_x] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $[P_y] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

5. **Reflections:** the map S_L defined by taking the symmetric of a vector v about a fixed line L is a linear transformation. They are invertible transformations, in fact each reflection is its own inverse. For example, $[S_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We should remark that the translations, i.e. $T(v) = v + v_0$, for some fixed v_0, are not linear transformations.

Proposition. *Any nonzero linear transformation of $\mathbb{R}^2$ can be obtained as a composition of linear transformations of types 1-5.*

Proof. The heart of the proof is the following decomposition of 2×2 invertible matrices (it's actually part of a more general result): any invertible 2×2 matrix A

can be written as a product

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

for some θ, a, b, n. If $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ has determinant one, set $\theta = \arctan(\frac{x}{z})$, $a = \sqrt{x^2+z^2}$, and $n = \frac{xy+yz}{x^2+z^2}$ (x and z cannot both be zero, since A is invertible). Verify by a direct calculation that the decomposition above (with $b = a^{-1}$) holds. The case of general invertible A follows easily, by multiplying the diagonal matrix with $det(A)$. We hope to treat this decomposition in a more conceptual way later in the course.

In this decomposition, the first matrix corresponds to a rotation, the diagonal matrix to a dilation $D_{a,b}$ or a composition of a dilation with a reflection (if a and b are not both positive), and the third matrix corresponds to a shear transformation. Therefore the invertible linear transformation with matrix A can be realized as a composition of a rotation, a reflection, a dilation, and a shear transformation.

Now, consider a general (maybe noninvertible) linear transformation with matrix $B \neq 0$. Then $B = UR$, where U is an invertible matrix and R is the row-reduced echelon form. We know the transformation corresponding to U can be decomposed as above, so it remains to analyze the transformations corresponding to all reduced echelon forms. If R is the identity (the case when B is invertible), there is nothing more left to do. Otherwise, R can only be of three forms $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix}$, with $s \neq 0$, or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The first form is clearly the projection P_x. The second form can be decomposed as

$$\begin{pmatrix} 1 & s \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

It means that the corresponding transformation is a composition between a projection and a shear transformation.

The third form can be decomposed as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so it is the composition of P_x with the reflection about the line $x = y$.

This concludes the proof. □