18.700 STUDY GUIDE FOR EXAM 3

Exam 3 will be in lecture, Monday, November 21. It will be closed book, closed notes, and calculators will not be allowed. You will have approximately 50 minutes for the exam. You should show all work, unless instructed otherwise; partial credit will be given only for work shown.

This guide contains a checklist of important skills, definitions and theorems to learn before Exam 3. You do not need to memorize the proofs of the theorems, but you should learn the statements and be able to use the theorems.

Skills checklist:

1. Given a linear transformation of vector spaces $T : V \to W$ and ordered bases $\mathcal{B}$ for V and $\mathcal{C}$ for W, compute the matrix representative of T with respect to B and C (3.3). Find the kernel and image of T. Know how to compute the transition matrix between two bases of the same space.
2. Compute the characteristic polynomial of a linear operator or a matrix (Def. 3.4.4 and 3.4.9), and (in simple cases) the minimal polynomial.
3. For a characteristic polynomial that has a simple factorization, find the eigenvalues of the linear operator (e.g., know the quadratic formula, and know how to find the rational zeros of a polynomial with integer coefficients).
4. Given a linear operator and an eigenvalue r, find the associated r-eigenspace and the generalized r-eigenspaces (e.g., Example 3.4.7 and the examples from Jordan canonical form).
5. Given a linear operator T decide if it is diagonalizable, and if it is find a basis $\mathcal{B}$ in which the associated matrix $[T]_{\mathcal{B}}$ is diagonal. Same problem for a matrix A: decide if it diagonalizable, and find the transition matrix P to the diagonal D: $A = PDP^{-1}$.
6. For small matrices or simple matrices (e.g. upper triangular with mostly zero entries) find the Jordan canonical form.

Definition checklist: Learn each of the following definitions.

linear operator (p.171); **kernel, image, one-to-one, onto, rank, nullity, isomorphism** (3.2); **matrix representative of a linear transformation** (Def. 3.3.1); **transition matrix, similar matrices** (3.4); **eigenvalue, eigenvector, eigenspace** (Def. 3.4.1); **characteristic polynomial** (Def. 3.4.4. and 3.4.9); **algebraic multiplicity, geometric multiplicity** (Def. 3.5.7); **diagonalizable operator** (Def. 3.5.1); **nilpotent matrix**; **minimal polynomial**; **generalized eigenspaces, Jordan blocks, Jordan canonical form** (class notes).

Theorem checklist: Learn the statements of the following theorems (you do not need to learn them verbatim, but learn their meaning). You do not need to memorize the proofs, but be able to use the theorems.

Theorem 3.2.8: characterization of one-to-one, onto linear transformations and isomorphisms.

Theorem 3.2.6 (Rank-Nullity) Suppose $T : V \to W$ is a linear transformation, where V is a finite-dimensional vector space. Then $rk(T) + null(T) = dim(V)$.

Theorem 3.3.6 Let $S : U \to V$ and $T : V \to W$ be linear transformations, let $\mathcal{A}$, resp. $\mathcal{B}$ and $\mathcal{C}$, be ordered bases for U, resp. V and W. Then $[T \circ S]_{\mathcal{C},\mathcal{A}} = [T]_{\mathcal{C},\mathcal{B}} \cdot [S]_{\mathcal{B},\mathcal{A}}$.

Corollary 3.3.9 Suppose that $T : V \to V$ is a linear transformation, where V is finite dimensional, and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ordered bases of V. Then $[T]_{\mathcal{B}_2} = P[T]_{\mathcal{B}_1}P^{-1}$, where P is the transition matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$.

Lemma 3.4.5 The eigenvalues of a linear operator T are the zeros of the associated characteristic polynomial $C_T(X)$.

Lemma 3.4.8 The characteristic polynomial of an operator T is independent of the choice of ordered basis.

Theorem 3.5.8 A linear operator T is diagonalizable iff both,

(i) the characteristic polynomial $C_T(X)$ factors as a product of linear factors, say $C_T(X) = (X - r_1)^{e_1} \dots (X - r_s)^{e_s}$ for distinct scalars $r_1, \dots, r_s$ and positive integers $e_1, \dots, e_s$, and

(ii) for every eigenvalue r_i the algebraic multiplicity e_i equals the geometric multiplicity $dim(E_{r_i})$.

Cayley-Hamilton Theorem For a linear operator T with characteristic polynomial $C_T(X)$, the operator $C_T(T)$ is the zero operator.

Theorem The minimal polynomial $M_T(X)$ divides the characteristic polynomial $C_T(X)$ and they have the same roots (except for multiplicities).

Theorem An operator T is diagonalizable if and only if its minimal polynomial factors into a product of distinct linear factors, e.g. $M_T(X) = (X - r_1) \dots (X - r_s)$, with $r_1, \dots, r_s$ all distinct.

Direct sum decomposition theorem Let $T : V \to V$ be a linear operator and assume $C_T(X)$ factors as $(X-r_1)^{e_1} \dots (X-r_s)^{e_s}$. If $E_{r_i}^{gen}$ denotes the generalized r_i-eigenspace of T, then $dim(E_{r_i}^{gen}) = e_i$ (in particular, the geometric multiplicity is less than or equal to the algebraic multiplicity), and V decomposes into a direct sum of the generalized eigenspaces of T: $V = E_{r_1}^{gen} \oplus \cdots \oplus E_{r_s}^{gen}$.

Jordan canonical form Let $T : V \to V$ be a linear operator and assume $C_T(X)$ factors as $(X - r_1)^{e_1} \dots (X - r_s)^{e_s}$. There exists an ordered basis for V with respect to which T is in Jordan canonical form.

Note that some of the above theorems for linear operators have equivalent formulations for (square) matrices.