18.700. Exam 3. Fall 2005.

Name: _____ November 21, 2005

Problem 1: _____ /30

Problem 2: _____ /30

Problem 3: _____ /25

Problem 4: _____ /15

Total: _____ /100

Instructions: The exam is closed book, closed notes and calculators are not allowed. You will have 50 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. Good luck!

Problem 1(30 points) Let V be the vector space of real-valued functions on $\mathbb{R}$ spanned by the four functions:

$$f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = x \sin x, \quad f_4(x) = x \cos x.$$

Let $\mathcal{B}$ be the ordered basis (f_1, f_2, f_3, f_4) of V and consider the linear transformation

$$T : V \to V, \qquad T(f(x)) = f(x + \pi) + f(-x).$$

Recall: $\sin(x + \pi) = -\sin x; \quad \cos(x + \pi) = -\cos x.$

(a) (10 points) Compute the matrix $[T]_{\mathcal{B}}$.

(b) (10 points) Show that T is diagonalizable and find a basis $\mathcal{C}$ of V such that $[T]_{\mathcal{C}}$ is diagonal.

(c) (5 points) Is T one-to-one?

(d) (5 points) Is the function $\cos x$ in the *image* of T?

Problem 2 (30 points) Consider the matrix

$$A = \begin{pmatrix} t & -3 \\ 3 & 0 \end{pmatrix}.$$

(a) (10 points) Find the values of the parameter $t \in \mathbb{R}$ for which A is *not* diagonalizable *over* $\mathbb{R}$.

(b) (10 points) $A = \begin{pmatrix} t & -3 \\ 3 & 0 \end{pmatrix}$. For $t = 6$, find the Jordan canonical form J of A and the transition matrix P such that $A = PJP^{-1}$.

(c) (10 points) Let B be a 2×2 matrix with real entries. Prove that if B doesn't have real eigenvalues, then B is diagonalizable over $\mathbb{C}$.

Problem 3 (25 points) Prove the following assertions.

(a) (9 points) Let A and B be $n \times n$ matrices such that $AB = A - B$. If 2 is an eigenvalue of B, then -2 is an eigenvalue of A.

(b) (8 points) If $N \neq 0$ is a nilpotent matrix, then N is *not* diagonalizable.

(c) (8 points) If the operator $T : V \to V$ is diagonalizable, then so is the operator $p(T)$, for any polynomial $p(X)$.

Problem 4 (15 points) Assume a matrix A has the Jordan form

$$J = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find the Jordan form of the matrix A^2.