EXAM 2

Instructions: You will have approximately 50 minutes for this exam. The test is closed book, closed notes and calculators are not allowed. The point value of each problem is written next to the problem – use your time wisely. Partial credit will be given. You may use either pencil or ink. If you need extra paper, raise your hand (I also have a stapler and paper clips for attaching extra sheets). If you have any questions, raise your hand. Please show all work unless instructed otherwise.

Problem 1(20 points) In each of the following cases, you are given a matrix with real coefficients. Calculate the determinant of each matrix. You do not need to show all steps, but do say what method you are using to calculate the matrix (row reduction, cofactor expansion, or by the equation using permutations).

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}. \tag{1}$$

Solution: Because this is an upper triangular matrix, the determinant is simply the product of the entries on the main diagonal, namely $1 \times 4 \times 6 = 24$.

$$B = \begin{pmatrix} 11 & 0 & 0 & 3 \\ 12 & 2 & 10 & 4 \\ 13 & 0 & 1 & 5 \\ 4 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Solution:

Here cofactor expansion works well. We first expand in the second column, since that has only one non-zero entry. This gives us that the determinant is $+2\det(B[2,2])$ where $B[i,j]$ is the matrix obtained by deleting the ith row and jth column. For $B[2,2]$ we notice that the second column (the third column in our original matrix) has only one non-zero entry. So $\det(B[2,2]) = +1\det((B[2,2])[2,2])$. Now $(B[2,2])[2,2]$ is a 2×2 matrix with determinant $11 \times 0 - 3 \times 4$. So, altogether we have $\det(B) = -1 \times 2 \times 3 \times 4 = -24$.

$$C = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 3 & 7 & -1 & 2 \\ 3 & 6 & 6 & 3 \end{pmatrix}. \tag{3}$$

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Solution: Here we notice that the fourth row is 3 times the second row. By n-linearity, we may factor the 3 out of the fourth row. By the alternating property, the resulting determinant is zero. Thus $\det(C) = 3 \times 0 = 0$.

Problem 2(15 points) Consider the 2×2 matrices with real entries:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}. \tag{4}$$

These are invertible matrices. Showing all work and explaining the steps, calculate $\det(A^\dagger B^\dagger A^{-1} B^{-1})$.

Solution: We use the properties of the determinant. First of all, for any 2×2 matrices C and D, we have $\det(CD) = \det(C)\det(D)$. Applying this three times, we conclude that

$$\det(A^\dagger B^\dagger A^{-1} B^{-1}) = \det(A^\dagger)\det(B^\dagger)\det(A^{-1})\det(B^{-1}). \tag{5}$$

Now we use the fact that for any square matrix C, we have $\det(C^\dagger) = \det(C)$. Applying this twice, we get

$$\det(A^\dagger)\det(B^\dagger)\det(A^{-1})\det(B^{-1}) = \det(A)\det(B)\det(A^{-1})\det(B^{-1}). \tag{6}$$

Next we use the fact that for any invertible matrix C, we have $\det(C^{-1}) = (\det(C))^{-1}$. Applying this twice, we get

$$\det(A)\det(B)\det(A^{-1})\det(B^{-1}) = \det(A)\det(B)\,(\det(A))^{-1}\,(\det(B))^{-1}. \tag{7}$$

But when we gather terms, the factors $\det(A)\det(A)^{-1}$ cancel to give 1, as do the factors $\det(B)\det(B)^{-1}$. Thus we finally conclude

$$\det(A^\dagger B^\dagger A^{-1} B^{-1}) = 1. \tag{8}$$

Problem 3(30 points) Consider the real vector space V of all real-valued functions of the form

$$f(x) = (a_1 + a_2 x)\sin(x) + (a_3 + a_4 x)\cos(x). \tag{9}$$

An ordered basis for V is

$$\mathcal{B} = (\sin(x), x\sin(x), \cos(x), x\cos(x)). \tag{10}$$

Recall the trigonometry identities

$$\cos\left(x + \frac{\pi}{2}\right) = -\sin(x), \sin\left(x + \frac{\pi}{2}\right) = \cos(x). \tag{11}$$

(a)(20 points) Consider the $\mathbb{R}$-linear transformation $D : V \to V$ defined by

$$D(f) = x\frac{df}{dx}(x) - xf\left(x + \frac{\pi}{2}\right). \tag{12}$$

You may assume that D is linear, you do not need to prove this. Calculate the matrix representative $[D]_{\mathcal{B},\mathcal{B}}$. You do not need to be rigorous, but please show all work.

Solution: Consider a function of the form $f(x) = g(x)\sin(x)$. Applying the Leibniz rule, we have

$$x\frac{d}{dx}\left(g(x)\sin(x)\right) - xg\left(x+\frac{\pi}{2}\right)\sin\left(x+\frac{\pi}{2}\right) = \tag{13}$$

$$xg'(x)\sin(x) + x\left[g(x) - g\left(x+\frac{\pi}{2}\right)\right]\cos(x). \tag{14}$$

If $g(x)$ is constant, then both the term $g'(x)$ and the term $g(x) - g(x+\frac{\pi}{2})$ are zero. If $g(x) = x$, then we get $D(x\sin(x)) = x\sin(x) - \frac{\pi}{2}x\cos(x)$.

Similarly, if $f(x) = g(x)\cos(x)$, we get the equation

$$D(f) = xg'(x)\cos(x) - x\left[g(x) - g\left(x+\frac{\pi}{2}\right)\right]\sin(x). \tag{15}$$

If $g(x)$ is constant, then both the term $g'(x)$ and the term $g(x) - g(x+\frac{\pi}{2}$ are zero. If $g(x) = x$, then we get $D(x\cos(x)) = x\cos(x) + \frac{\pi}{2}x\sin(x)$.

Taking coordinate vectors with respect to our basis, we conclude that the matrix representative is

$$[D]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\pi}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\pi}{2} & 0 & 1 \end{pmatrix}. \tag{16}$$

(b)(10 points) Find all solutions in the vector space V of the differential equation

$$x\frac{df}{dx}(x) - xf\left(x+\frac{\pi}{2}\right) - f(x) = \frac{3\pi x}{2}\cos(x). \tag{17}$$

You do not need to be rigorous, but please show all work.

Solution: If we translate into matrix form, the equation is

$$\left([D]_{\mathcal{B},\mathcal{B}} - I_4\right)(f)_{\mathcal{B}} = \left(\frac{3\pi x}{2}\cos(x)\right)_{\mathcal{B}}. \tag{18}$$

Substituting in, this becomes the matrix equation

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\pi}{2} \\ 0 & 0 & -1 & 0 \\ 0 & -\frac{\pi}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{3\pi}{2} \end{pmatrix}. \tag{19}$$

Notice that the matrix has determinant $\frac{\pi^2}{4} \neq 0$. Thus, if there is a solution, there is a unique solution. From the form of the matrix, it is easy to guess the solution (or to use Gauss-Jordan elimination to find the solution):

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

So we see that the unique solution in V is $f(x) = 3x\sin(x)$.

Problem 4 Recall that a *permutation of* n is an ordered n-tuple $\sigma = (i_1, i_2, \ldots, i_n)$ where $i_1, i_2, \ldots, i_n$ are distinct integers with $1 \leq i_j \leq n$ for each $j = 1, \ldots, n$. Recall that the *sign* of a permutation σ is the number $\text{sgn}(\sigma)$ of *inversions*, i.e. pairs (j, k) such that $1 \leq j < k \leq n$ and also $i_k < i_j$.

(a)(15 points) Suppose that $\sigma = (i_1, i_2, i_3)$ and $\sigma' = (i'_1, i'_2, i'_3)$ are two permutations of 3. Form the permutation of 6 which is $\tau = (i_1, i_2, i_3, 3 + i'_1, 3 + i'_2, 3 + i'_3)$. Prove that

$$\text{sgn}(\tau) = \text{sgn}(\sigma)\text{sgn}(\sigma'). \tag{21}$$

Show all work and justify your answer.

Solution: Suppose that after applying a sequence of r transpositions, say (a_j, b_j), then σ has the form $(1, 2, 3)$. Suppose that after applying a sequence of s transpositions, say (c_k, d_k), then σ' has the form $(1, 2, 3)$. Then after first applying the r transpositions (a_j, b_j) and then applying the s transpositions $(3 + c_j, 3 + d_j)$, τ will have the form $(1, 2, 3, 4, 5, 6)$. By the axioms for sgn, we know that $\text{sgn}(\sigma) = (-1)^r, \text{sgn}(\sigma') = (-1)^s$ and $\text{sgn}(\tau) = (-1)^{r+s}$. Thus we conclude that $\text{sgn}(\tau) = \text{sgn}(\sigma)\text{sgn}(\sigma')$.

(b)(20 points) Suppose that $A = (a_{i,j})$ and $B = (b_{i,j})$ are two 3×3 matrices. Consider the 6×6 matrix

$$C = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{1,1} & b_{1,2} & b_{1,3} \\ 0 & 0 & 0 & b_{2,1} & b_{2,2} & b_{2,3} \\ 0 & 0 & 0 & b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}. \tag{22}$$

By whatever (rigorous) method you like, prove that $\det(C) = \det(A)\det(B)$. (HINT: Permutations and determinants).

Solution: Recall the formula for the determinant of a 6×6 matrix $C = (c_{i,j})$ is

$$\det(C) = \sum_{\tau \in S_6} \text{sgn}(\tau) c_{1,i_1} \ldots c_{6,i_6}. \tag{23}$$

Now from the form of C above, we see that $c_{i,j} = 0$ unless we are one of the two cases $1 \leq i, j \leq 3$ or $4 \leq i, j \leq 6$. So the only monomials in the formula of determinant which are nonzero are those enumerated by permutations $\tau = (i_1, \ldots, i_6)$ such that $1 \leq i_j \leq 3$ for $1 \leq j \leq 3$ and $4 \leq i_j \leq 6$ for $4 \leq j \leq 6$. In this case we see that $\tau = (i_1, i_2, i_3, 3 + i'_1, 3 + i'_2, 3 + i'_3)$ for unique permutations $\sigma = (i_1, i_2, i_3)$ and $\sigma' = (i'_1, i'_2, i'_3)$ of 3. So, applying part (a), we conclude that

$$\det(C) = \sum_{\sigma \in S_3} \sum_{\sigma' \in S_3} \text{sgn}(\sigma)\text{sgn}(\sigma') a_{1,i_1} a_{2,i_2} a_{3,i_3} b_{1,i'_1} b_{2,i'_2} b_{3,i'_3}. \tag{24}$$

But using associativity and commutativity of addition and distributivity of multiplication and addition, the right hand side is simply

$$\left(\sum_{\sigma\in S_3} \text{sgn}(\sigma) a_{1,i_1}a_{2,i_2}a_{3,i_3}\right)\left(\sum_{\sigma'\in S_3} \text{sgn}(\sigma') b_{1,i_1}b_{2,i_2}b_{3,i_3}\right) = \det(A)\det(B). \tag{25}$$

So we conclude that $\det(C) = \det(A)\det(B)$.

EXTRA CREDIT Please only attempt the extra credit after you have finished the exam and checked your answers.

(a)(5 points) Suppose $\sigma = (i_1, i_2, i_3)$ and $\sigma' = (i'_1, i'_2, i'_3)$ are two permutations of 3. Consider the permutation of 6, $\tau = (3+i'_1, 3+i'_2, 3+i'_3, i_1, i_2, i_3)$. Prove that $\text{sgn}(\tau) = -\text{sgn}(\sigma)\text{sgn}(\sigma')$.

Solution: With the same notation as in (a) above, after performing the transpositions $(3+a_i, 3+b_i)$ and (c_i, d_i), we have that τ is in the form $(4, 5, 6, 1, 2, 3)$. Now we perform the three permutations $(1, 4), (2, 5), (3, 6)$ and we get $(1, 2, 3, 4, 5, 6)$. Therefore $\text{sgn}(\tau) = (-1)^{r+s+3} = -\text{sgn}(\sigma)\text{sgn}(\tau)$.

(b)(5 points) Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two 3×3 matrices. Define C to be the 6×6 matrix:

$$C = \begin{pmatrix} 0 & 0 & 0 & a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & 0 & a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & 0 & 0 & a_{3,1} & a_{3,2} & a_{3,3} \\ b_{1,1} & b_{1,2} & b_{1,3} & 0 & 0 & 0 \\ b_{2,1} & b_{2,2} & b_{2,3} & 0 & 0 & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

By whatever (rigorous) method you like, prove that $\det(C) = -\det(A)\det(B)$.

Solution: This is similar to case (b) above, except that now the monomial in the determinant is nonzero only when $\tau = (j_1, j_2, j_3, j_4, j_5, j_6)$ satisfies $4 \le j_k \le 6$ for $1 \le k \le 3$ and $1 \le j_k \le 3$ for $4 \le k \le 6$. Every such permutation is of the form $(3+i'_1, 3+i'_2, 3+i'_3, i_1, i_2, i_3)$ for unique permutations $\sigma = (i_1, i_2, i_3)$ and $\sigma' = (i'_1, i'_2, i'_3)$ of 3. Therefore the formula for the determinant becomes

$$\det(C) = \sum_{\sigma\in S_3}\sum_{\sigma'\in S_3} -\text{sgn}(\sigma)\text{sgn}(\sigma') a_{1,i_1}a_{2,i_2}a_{3,i_3}b_{1,i'_1}b_{2,i'_2}b_{3,i'_3}. \tag{27}$$

Simplifying this expression, we get

$$-1\left(\sum_{\sigma\in S_3} \text{sgn}(\sigma) a_{1,i_1}a_{2,i_2}a_{3,i_3}\right)\left(\sum_{\sigma'\in S_3} \text{sgn}(\sigma') b_{1,i_1}b_{2,i_2}b_{3,i_3}\right) = -\det(A)\det(B). \tag{28}$$