18.700. Exam 2. Fall 2005. Solutions

Problem 1(33 points) For the following matrices, compute the determinant, *in (a) and (b), by the method indicated*. Show work.

(a)(11 points) Using cofactor expansion:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \\ 1 & 0 & 9 & 5 \end{pmatrix}.$$

-28.

(b)(11 points) Using row-reduction:

$$B = \begin{pmatrix} 1 & 0 & 0 & 9 & 8 \\ 2 & 1 & 0 & 7 & 6 \\ 3 & 2 & 1 & 5 & 3 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 5 & 3 \end{pmatrix}.$$

1.

(c)(11 points)

$$M_n = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & 3 & \dots & 3 \\ 1 & 2 & 3 & 4 & \dots & 4 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & 4 & \dots & n \end{pmatrix}$$

(the ijth entry of M_n is the minimum of i and j, n is arbitrary, positive integer).

There are two ways to compute the determinant by row-reduction. The first method is to apply row operations $R_i \to R_i - R_{i-1}$ starting with the nth row and moving up. In this way, one ends up with an upper triangular matrix with 1's everywhere, including the diagonal. Therefore the determinant is 1.

Alternatively, substract the first row from all other rows and then expand with respect to the first column. In this way, $detM_n = detM_{n-1}$, and by induction, since $detM_1 = 1$), it follows that $detM_n = 1$.

Problem 2 (30 points)
(a)(15 points) Using the formula for the inverse in terms of the adjoint, find the inverse of the matrix

$$A = \begin{pmatrix} \cos(\theta) & 1 & -\sin(\theta) \\ 0 & 2 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

Using cofactor expansion about the second row, it follows that $detA = 2\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = 2.$
$A^{-1} = (detA)^{-1}adj(A)$. The j,i-entry in the adjoint is $adj(A)(j,i) = (-1)^{i+j}detA(i|j)$. Computing the determinants of all maximal submatrices, it follows that $A^{-1} = \begin{pmatrix} \cos(\theta) & -\frac{1}{2}\cos(\theta) & \sin(\theta) \\ 0 & \frac{1}{2} & 0 \\ -\sin(\theta) & \frac{1}{2}\sin(\theta) & \cos(\theta) \end{pmatrix}.$

(b) (15 points) Using Cramer's rule, find the solution of the system

$$\begin{cases} 2X_1 + X_2 = 1 \\ X_1 + 2X_2 + X_3 = 0 \\ X_2 + 2X_3 = 0 \end{cases}$$

(the determinant of the matrix associated to the system is 4)

The matrix associated to the system is $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Note that $detA = 4$. By Cramer's rule, the solution is:

$$X_1 = \frac{1}{detA}det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \frac{3}{4}.$$

$$X_2 = \frac{1}{detA}det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = -\frac{2}{4}.$$

$$X_3 = \frac{1}{detA}det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{4}.$$

Problem 3(27 points) Let P^n be the real vector space of polynomials in the variable X of degree $\leq n$, with real coefficients.

(a) (15 points) Consider the following two bases of P^2: $\mathcal{B} = \{1, X, X^2\}$ and $\mathcal{C} = \{X+1, X-1, X^2+2X\}$. Find the change of basis matrix A such that

$$(f)_{\mathcal{C}} = A \cdot (f)_{\mathcal{B}},$$

for any polynomial $f \in P^2$.

A is the change of basis matrix from the basis $\mathcal{C}$ to the basis $\mathcal{B}$. We express that polynomials in $\mathcal{B}$ as linear combinations of the polynomials in $\mathcal{C}$.

$1 = \frac{1}{2}(X+1) - \frac{1}{2}(X-1)$.

$X = \frac{1}{2}(X+1) + \frac{1}{2}(X-1)$.

$X^2 = -(X+1) - (X-1) + (X^2+2X)$.

Therefore, $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) (12 points) Let $T : P^n \to P^n$ be the linear transformation given by

$$T(f(X)) = Xf'(X) + f(X).$$

Find the matrix associated to T with respect to the standard basis of P^n.

The standard basis of P^n is $\{1, X, X^2, \ldots, X^n\}$. For each monomial X^j, $T(X^j) = (j+1)X^j$, $j = 0, n$. The matrix $[T]$ is a $(n+1) \times (n+1)$ diagonal matrix then with $1, 2, 3, \ldots, n+1$ on the diagonal.

Problem 4(10 points) Let A be an $n \times n$ matrix, with odd integer entries on the diagonal and even integer entries everywhere else. Show that A must be invertible.

The easiest way to prove this is by the formula for the determinant in terms of the permutations. $det A = \sum_{\sigma \in S_n} sg(\sigma) a_{1\sigma_1} a_{2\sigma_2} \dots a_{n\sigma_n}$.
The term corresponding to the identity permutation is the product of the diagonal entries, so it is odd. Any other term will contain off-diagonal terms, which are even, and therefore, any other term in the sum is even. This argument implies that $det A$ is an odd integer, so nonzero. A is invertible.