18.700. Exam 1. Fall 2005. Solutions

Problem 1(40 points) Let A be the 3×5 matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & 1 & 3 \\ 1 & 0 & 1 & 2 & -1 \\ 3 & 1 & 2 & 5 & -2 \end{pmatrix}.$$

a)(20 points) Find all its right inverses, if they exist.
Row reduce the augmented matrix

$$\begin{pmatrix} 2 & 1 & -1 & 1 & 3 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & -1 & | & 0 & 1 & 0 \\ 3 & 1 & 2 & 5 & -2 & | & 0 & 0 & 1 \end{pmatrix}.$$

The row reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & | & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & | & -\frac{1}{2} & -\frac{7}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1 & -2 & | & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The most general right inverse is

$$\begin{pmatrix} \frac{1}{2} - s_1 - t_1 & \frac{3}{2} - s_2 - t_2 & -\frac{1}{2} - s_3 - t_3 \\ -\frac{1}{2} + t_1 & -\frac{7}{2} + t_2 & \frac{3}{2} + t_3 \\ -\frac{1}{2} - s_1 + 2t_1 & -\frac{1}{2} - s_2 + 2t_2 & \frac{1}{2} - s_3 + 2t_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix},$$

for any real $s_1, t_1, s_2, t_2, s_3, t_3$.

(b)(10 points) Find a basis for the column space of A and write the fifth column in coordinates with respect to this basis.

From the reduced echelon form on A, we see that the first three columns have the leading ones, so the basis for the column space is $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$. From the fifth column in the reduced form we conclude that, in this basis, $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}_{\mathcal{B}}$.

(c)(10 points) Find a basis for the nullspace of A.

Again using the reduced form of A, we see that the free variables in the reduced homogeneous system correspond to the column 4 and 5. A basis of the nullspace is $\left\{ \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Problem 2(32 points)

(a)(10 points) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$. First compute AB, then BA.

$AB = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 4 & -1 & 2 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 2 & -1 & 2 \end{pmatrix}$.

(b)(10 points) Assume A and B are some 2×2 matrices such that $AB - BA = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$. Find $A^tB^t - B^tA^t$.

$A^tB^t - B^tA^t = (BA)^t - (AB)^t = (BA - AB)^t = -(AB - BA)^t = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$.

(c)(12 points) For the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, determine all 3×3 matrices B such that $AB = BA$.

Let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. Then $AB = \begin{pmatrix} 2b_{11} & 2b_{12} & 2b_{13} \\ 2b_{21} & 2b_{22} & 2b_{23} \\ -b_{31} & -b_{32} & -b_{33} \end{pmatrix}$ and $BA = \begin{pmatrix} 2b_{11} & 2b_{12} & -b_{13} \\ 2b_{21} & 2b_{22} & -b_{23} \\ 2b_{31} & 2b_{32} & -b_{33} \end{pmatrix}$.

Then $AB = BA$ if and only if $b_{13} = b_{23} = b_{31} = b_{32} = 0$. The matrices which commute with A are all matrices of the form $\begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$.

Problem 3(28 points) Answer the following questions "True" or "False". *Give* clear and concise explanations of your answers or *show* counterexamples.

(a)(7 points) If a vector space V is n-dimensional, then every subset of V with more than n elements is a spanning set for V.

False. Counterexample: $V = \mathbb{R}^2$ is 2-dimensional over $\mathbb{R}$, but the set $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$ is not a spanning set for V. The key observation is that a set with more than n elements may not contain n linearly independent vectors.

(b)(7 points) The set of polynomials $\{f_1(X), f_2(X), \ldots, f_r(X)\}$ is linearly independent *if* the set of polynomials $\{Xf_1(X), Xf_2(X), \ldots, Xf_r(X)\}$ is linearly independent.

True. Let $a_1f_1(X) + a_2f_2(X) + \cdots + a_rf_r(X) = 0$ be a linear relation (here) means the zero polynomial). We want to conclude that necessarily $a_1 = a_2 = \cdots = a_r = 0$. Then $a_1Xf_1(X) + a_2Xf_2(X) + \cdots + a_rXf_r(X) = X \cdot 0 = 0$. Since $\{Xf_1(X), Xf_2(X), \ldots, Xf_r(X)\}$ are linearly independent, necessarily $a_1 = a_2 = \cdots = a_r = 0$, which concludes the proof.

(c)(7 points) Let v be a fixed vector in $\mathbb{R}^n$. The set of $m \times n$ real matrices A with the property that $Av = 0$ form a real vector space.

True. Note that the set in question is not the nullspace. We verify the axioms for the vector (sub)space. The zero matrix 0 is in the set, since $0 \cdot v = 0$. If two matrices A_1 and A_2 have the property that $A_1v = A_2v = 0$, then $(A_1 + A_2)v = 0 + 0 = 0$, so $A_1 + A_2$ is in this set. Also if $AV = 0$ and k is a real scalar, $(kA)v = k(Av) = k(0) = 0$, so kA is also in the set.

(d)(7 points) (Recall that $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ is the field with two elements) There exists a vector space V over $\mathbb{F}_2$ with exactly 18 elements.

False. One of the corollaries to "coordinates with respect to a basis" was that any n-dimensional $\mathbb{F}$-vector space is similar to $\mathbb{F}^n$. Since $\mathbb{F}_2$ here has 2 elements, we conclude that an n-dimensional $\mathbb{F}_2$-vector space must have 2^n elements. But 18 is not a power of 2.