18.700. Exam 3. Fall 2005.

Problem 1(30 points) Let V be the vector space of real-valued functions on $\mathbb{R}$ spanned by the four functions:

$$f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = x \sin x, \quad f_4(x) = x \cos x.$$

Let $\mathcal{B}$ be the ordered basis (f_1, f_2, f_3, f_4) of V and consider the linear transformation

$$T : V \to V, \qquad T(f(x)) = f(x + \pi) + f(-x).$$

Recall: $\sin(x + \pi) = -\sin x; \quad \cos(x + \pi) = -\cos x.$

(a) (10 points) Compute the matrix $[T]_{\mathcal{B}}$.

We also use that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.
$T(\sin x) = -2\sin x;$
$T(\cos x) = 0;$
$T(x \sin x) = -\pi x \sin x;$
$T(x \cos x) = -\pi \cos x - 2x \cos x.$
Then $[T]_{\mathcal{B}} = \begin{pmatrix} -2 & 0 & -\pi & 0 \\ 0 & 0 & 0 & -\pi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

(b) (10 points) Show that T is diagonalizable and find a basis $\mathcal{C}$ of V such that $[T]_{\mathcal{C}}$ is diagonal.

T is diagonalizable if and only if $[T]_{\mathcal{B}}$ is diagonalizable. $C_T(X) = (X + 2)^2 X^2$, so the eigenvalues are -2 and 0, both with algebraic multiplicity 2.
We compute the eigenspaces:

$$E_{-2} = ker(-2I - [T]_{\mathcal{B}}) = ker \begin{pmatrix} 0 & 0 & \pi & 0 \\ 0 & -2 & 0 & \pi \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = ker \begin{pmatrix} 0 & 1 & 0 & -\frac{\pi}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$span\left\{ \begin{pmatrix} 0 \\ \frac{\pi}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$E_0 = ker(-[T]_{\mathcal{B}}) = ker\begin{pmatrix}2 & 0 & \pi & 0\\ 0 & 0 & 0 & \pi\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} = ker\begin{pmatrix}1 & 0 & 0 & \frac{\pi}{2}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} =$$

$$span\{\begin{pmatrix}-\frac{\pi}{2}\\ 0\\ 1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix}\}.$$

Since the geometric multiplicities equal the algebraic multiplicities, T is diagonalizable. An eigenbasis is $\mathcal{C} = \{(x+\frac{\pi}{2})\cos x, \sin x, (x-\frac{\pi}{2})\sin x, \cos x\}$, and $[T]_{\mathcal{C}}$ is the diagonal matrix $(-2,-2,0,0)$.

(c) (5 points) Is T one-to-one?

The kernel of T is E_0, which has dimension 2, therefore $kerT \neq \{0\}$, and T is not one-to-one.

(d) (5 points) Is the function $\cos x$ in the *image* of T?

The image of T is spanned by $\{(x+\frac{\pi}{2})\cos x, \sin x\}$ and it is impossible to get $\cos x$ as a linear combination of these functions, therefore, $\cos x \notin imT$.

Problem 2 (30 points) Consider the matrix

$$A = \begin{pmatrix}t & -3\\ 3 & 0\end{pmatrix}.$$

(a) (10 points) Find the values of the parameter $t \in \mathbb{R}$ for which A is *not* diagonalizable *over* $\mathbb{R}$.

We compute the characteristic polynomial: $C_A(X) = X^2 - tX + 9$, which has discriminant $t^2 - 36$. If $|t| < 6$, $C_A(X)$ doesn't have real roots, so A can't be diagonalizable over $\mathbb{R}$. If $|t| > 6$, $C_A(X)$ has two distinct real roots, so A is diagonalizable.

It remains to analyze $t = \pm 6$. In this case A has a single eigenvalue with multplicity 2. We know that such a matrix could be diagonalizable only if it is already diagonal, so A is not diagonalizable.

The answer is: $-6 \leq t \leq 6$.

(b) (10 points) $A = \begin{pmatrix}t & -3\\ 3 & 0\end{pmatrix}$. For $t = 6$, find the Jordan canonical form J of A and the transition matrix P such that $A = PJP^{-1}$.

If $t = 6$, $C_A(X) = t^2 - 6t + 9$, and it has a single eigenvalue $r = 3$, with algebraic multplicity 2. The eigenspace $E_3 = ker(3I - A) = span\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$.

Since the geometric multiplicity of 3 is less than the algebraic multiplicity, A is not diagonalizable. The only possibility then is $J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$.

The transition matrix P is $P = (v_1|v_2)$, where $v_1 \in E_3$, and v_2 is such that $(A - 3I)v_2 = v_1$. We find for example $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $v_2 = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$, so $P = \begin{pmatrix} 1 & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$.

(c) (10 points) Let B be a 2×2 matrix with real entries. Prove that if B doesn't have real eigenvalues, then B is diagonalizable over $\mathbb{C}$.

Since the characteristic polynomial must have real coefficients, by the quadratic formula, the eigenvalues of B are of the form $a + bi$ and $a - bi$, with $b \neq 0$. Since B is 2×2 and has two distinct eigenvalues (geom. mult=alg. mult.=1), B must be diagonalizable.

Problem 3 (25 points) Prove the following assertions.

(a) (9 points) Let A and B be $n \times n$ matrices such that $AB = A - B$. If 2 is an eigenvalue of B, then -2 is an eigenvalue of A.

Let $v \neq 0$ be an eigenvector of B with $Bv = 2v$. Then $ABv = (A - B)v$ implies that $2Av = Av - 2v$, so $Av = -2v$, which says -2 is an eigenvalue of A.

(b) (8 points) If $N \neq 0$ is a nilpotent matrix, then N is *not* diagonalizable.

If N is nilpotent, by definition, there exists $k > 0$ such that $N^k = 0$. Assume N is diagonalizable, i.e., there exists P invertible and D diagonal, such that $N = PDP^{-1}$. Note that $D \neq 0$, since $N \neq 0$. Then $0 = N^k = PD^kP^{-1}$, so $D^k = 0$, contradiction.

(c) (8 points) If the operator $T : V \longrightarrow V$ is diagonalizable, then so is the operator $p(T)$, for any polynomial $p(X)$.

Let $\mathcal{B}$ be a basis of V such that $[T]_{\mathcal{B}}$ is diagonal. If $p(T) = a_0 I + a_1 T + \cdots + a_n T^n$, $[p(T)]_{\mathcal{B}} = a_0 I_n + a_1 [T]_{\mathcal{B}} + a_2 [T^2]_{\mathcal{B}} + \cdots + a_n [T^n]_{\mathcal{B}} = a_0 I_n + a_1 [T]_{\mathcal{B}} + a_2 [T]_{\mathcal{B}}^2 + \cdots + a_n [T]_{\mathcal{B}}^n$, so $[p(T)]_{\mathcal{B}}$ is also diagonal.

Note that we used the properties of matrices associated to linear transformations: $[S+T]=[S]+[T]$ and $[S\circ T]=[S]\cdot[T]$.

Problem 4 (15 points) Assume a matrix A has the Jordan form

$$J=\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find the Jordan form of the matrix A^2.

There exists an invertible matrix P such that $A=PJP^{-1}$. Then $A^2=PJ^2P^{-1}$, so A^2 is similar to J^2. We compute J^2 directly, $J^2 = \begin{pmatrix} 1 & -2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1 \end{pmatrix}$. Note that J^2 is not a Jordan form, but that the Jordan form of J^2 is also the Jordan form of A^2.

For J^2, $C(X)=(X-1)^4$, $dimE_1=2$, so there are two Jordan blocks, and $M(X)=(X-1)^2$, so the largest size of a block is 2. Therefore, the Jordan form is $\begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 \end{pmatrix}$.