18.700. Final Exam. Fall 2005. Solutions. December 22, 2005

Problem 1. a) Form the augmented matrix $\begin{pmatrix} -1 & 0 & 2 & | & 1 \\ 3 & -1 & 3 & | & 5 \\ 2 & 3 & -1 & | & -2 \end{pmatrix}$ and row reduce it. The reduced (upper triangular) form is $\begin{pmatrix} 1 & 0 & -2 & | & -1 \\ 0 & 1 & -9 & | & -8 \\ 0 & 0 & 1 & | & -\frac{4}{5} \end{pmatrix}$. This implies the system is consistent and a solution is obtain by back-substitution

$$X = \frac{3}{5},\ Y = -\frac{4}{5},\ Z = \frac{4}{5}.$$

b) Again we reduce the augmented matrix

$$\begin{pmatrix} 2 & 0 & 5 & -1 & | & 9 \\ 1 & -1 & 1 & 2 & | & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \frac{5}{2} & -\frac{1}{2} & | & \frac{9}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{5}{2} & | & \frac{7}{2} \end{pmatrix}.$$

The free variables are Z and W, and the most general solution is

$$\begin{pmatrix} \frac{9}{2} \\ \frac{7}{2} \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ 0 \\ 1 \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

Problem 2. a) The row reduced echelon form of A is

$$R = \begin{pmatrix} 1 & 0 & 2 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

b) A basis of the column space is given by the columns of A which correspond to the leading 1's in R, so the first, second and fourth columns. A basis of the row space of A is given by the rows containing the leading 1's in R. Recall that A and R have the same row space, but not the same column space in general (note that the particular matrix A in this problem is a bad choice for illustrating this, since the rows are independent and the column space is all of $\mathbb{R}^3$; I apologize).

c) The orthogonal complement of the row space of a matrix in $\mathbb{R}^n$ is the nullspace of the matrix. We already have the row reduced form of A, so a basis for the nullspace is $\{ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \}$.

Problem 3. Since the 3×5 matrix A has rank 3, it has right inverses B, i.e., 5×3 matrices such that $AB = I_3$. To find such a B, we form the matrix $\begin{pmatrix} 0 & 0 & 1 & 1 & 2 & | & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & -1 & | & 0 & 1 & 0 \\ 3 & -1 & 3 & 1 & -1 & | & 0 & 0 & 1 \end{pmatrix}$ and row reduce it to the echelon form. This is $\begin{pmatrix} 1 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & | & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -1 & | & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 3 & | & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. The second and fifth columns don't have leading 1's (there are two parameters in the most general right inverse). A particular right inverse is then

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 4. a) We need to find a maximal subset of linearly independent vectors (polynomials) of $\mathcal{B}$. We write each polynomial in $\mathcal{B}$ in coordinates with respect to the usual basis of P_4 and write them as columns in a matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. Then we need to find a basis for the column space, so we row reduce etc. It follows that $\mathcal{C} = \{v_1, v_2, v_3\} \subset \mathcal{B}$ is such a maximal independent subset.

b) In coordinates with respect to $\mathcal{C}$, $[v_1] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[v_2] = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $[v_3] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $[v_4] = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

c) $\mathcal{C}$ is a set of three independent vectors in P_4, so by the basis extension theorem, we can find two vectors in P_4 that complete $\mathcal{C}$ to a basis of P_4 ($dim(P_4) = 5$). For example $w_1 = 1$ and $w_2 = X^3$ work.

Problem 5. By your favourite method, the determinant is 7 (for example, one can expand about the second row and then compute the resulting 3×3 determinants).

Problem 6. a) A is 5×5 with even integer entries. By the permutation formula for the determinant, we see that $det(A)$ is an alternating sum of terms indexed by permutations of 5, such that every term in the sum is a product of 5 even integers. Therefore, each term is divisible by 32, and so $det(A)$ must be divisible by 32. So there are no such matrices with determinant 120.

b) Assume the determinant of A is 160 (note that this is divisible by 32). Then $A^{-1} = \frac{1}{det(A)} adj(A)$. Every entry in $adj(A)$ is $\pm$ a determinant of a 4×4 matrix with all entries even integers, so by a), 16 divides every entry in the $adj(A)$. It follows that the largest possible denominator in A^{-1} is 10. An example in which this maximum is attained is for the diagonal matrix A with entries $10, 2, 2, 2, 2$.

Problem 7. Compute $T(v_i)$, $i = 1, 3$ and write the results as linear combinations of $\{w_1, w_2, w_3\}$, e.g.,

$$T(v_1) = \big((e^x + e^{-x})\cos(x)\big)' = (e^x - e^{-x})\cos(x) - (e^x + e^{-x})\sin(x) = w_1 - w_3.$$

The associated matrix $[T]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

Problem 8. The characteristic polynomial of A is $C_A(X) = (X-2)^2(X+1)$, so the eigenvalues of A are 2 (algebraic multiplicity 2) and -1 (algebraic multiplicity 1).

The two eigenspaces are $E_{-1} = span\{\begin{pmatrix} -\frac{4}{3} \\ 1 \\ 3 \end{pmatrix}\}$, and $E_2 = span\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$. Since the geometric multiplicity of 2 is 1, it follows that there is single Jordan block with eigenvalue 2, so the Jordan form is $J = \begin{pmatrix} 2 & 1 & 0 & \\ 0 & 2 & 0 & \\ & 0 & 0 & -1 \end{pmatrix}$. To find a transition matrix P, denote the columns of P by v_1, v_2, respectively v_3. Then v_1 is an eigenvector in E_2, v_2 is a generalized eigenvector with eigenvalue 2, such that $(A-2I)v_2 = v_1$, and v_3 is an eigenvector in E_{-1}.

We compute $E_2^{(2)} = ker(A-2I)^2 = span\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$. Since $v_2 \in E_2^{(2)} \setminus E_2$, we can choose $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and so $v_1 = (A-2I)v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

A transition matrix P is

$$P = \begin{pmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Problem 9. Since S is real symmetric, it is diagonalizable and it has real eigenvalues. Moreover, the transition matrix Q can be chosen to be orthogonal, that is $S = QDQ^t$. The characteristic polynomial of A is $C_a(X) = X(X-3)^2$, so necessarily (up to reordering the eigenvalues) $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The eigenspaces are $E_3 = span\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\}$, and $E_0 = span\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$. Note that the two eigenspaces are automatically orthogonal (as they should be), so we only need to construct orthonormal bases for each eigenspace. For E_3, apply Gram-Schmidt to the basis vectors and normalize them. Then

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Problem 10. To find the QR-decomposition of A, we apply the Gram-Schmidt algorithm to A (note that A is invertible), and keep track of the coefficients which appear in the (recursive) formulas.

Denote by v_1, v_2, v_3 the columns of A, w_1, w_2, w_3 the unnormalized vectors obtained in the Gram-Schmidt algorithm, and by u_1, u_2, u_3 the corresponding normalized vectors. u_1, u_2, u_3 are the columns of Q.

$v_1 = (1, 0, 2)$. $w_1 = v_1$ and $u_1 = \frac{1}{\sqrt{5}}(1, 0, 2)$. Then $v_1 = \sqrt{5}u_1$.

$w_2 = v_2 - (v_2.u_1)u_1 = v_2 - \frac{3}{\sqrt{5}}u_1 = (\frac{2}{5}, 1, -\frac{1}{5})$ and $u_2 = \frac{\sqrt{30}}{6}w_2 = \frac{1}{\sqrt{30}}(2, 5, -1)$. Then $v_2 = \frac{3}{\sqrt{5}}u_1 + \frac{6}{\sqrt{30}}u_2$.

$w_3 = v_3 - (v_3.u_1)u_1 - (v_3.u_2)u_2 = v_3 + \frac{5}{\sqrt{30}}u_2 = (\frac{2}{6}, -\frac{1}{6}, -\frac{1}{6})$, and $u_3 = \sqrt{6}w_3 = \frac{1}{\sqrt{6}}(2, -1, -1)$. Then $v_3 = -\frac{5}{\sqrt{30}}u_2 + \frac{1}{\sqrt{6}}u_3$.

In matrix form

$$Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} & 0 \\ 0 & \frac{6}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}.$$