18.701 Problem Set 11

1. Describe the conjugacy classes in SO_3 in two ways:

(i) The elements operate on $\mathbb{R}^3$ as rotations. Decide which rotations make up a conjugacy class.

(ii) Using the spin homomorphism $SU_2 \to SO_3$, one can relate the conjugacy classes in these two groups. Do so, and using this relationship, describe the conjugacy classes of SO_3 as geometric loci. (Be careful: there is more than one possibility.)

2. Chapter 8, Problem 5.10.

3. Chapter 8, Problem 6.11. (There is a typo in the formula.)

4. Consider an equation $x^2 + cxy + y^2 = 1$, where c is a given element of a field F. We'll call the locus of this equation a "conic", and denote it by C. The points of C are the solutions $x = a$, $y = b$, with a, b in the field F.

(i) Compute the number of points (a, b) on every conic C directly, when $F = \mathbb{F}_5$.

(ii) (The following analysis is based on geometry. Don't be afraid to draw a picture.) If $c \neq \pm 2$, the points on C can be parametrized by lines through $(0, 1)$. Let L denote the line $y = \lambda x + 1$. For most values of λ, the intersection $C \cap L$ contains $(0, 1)$ and exactly one other point (a, b). Compute this second point explicitly in terms of λ.

(iii) The correspondence between points on the conic and lines through $(0, 1)$ isn't quite bijective. For one thing, a denominator in the formula for (a, b), might vanish for some λ. Whether it can vanish depends on c and F. Describe the errors in the correspondence, assuming that $c \neq \pm 2$. What happens when $c = \pm 2$?

(iv) Determine the number of points on C when $F = \mathbb{F}_p$, treating the cases $c = \pm 2$ directly. Check for agreement with (i).

5. (i) Determine the class equation for the group $G = SL_2(F)$, when $F = \mathbb{F}_p$ and $p \neq 2$, basing your analysis on characteristic polynomials. I suggest that you begin by determining the centralizer of the element

$$A = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}.$$

(ii) Use the class equation to prove that the only proper normal subgroup of $SL_2(\mathbb{F}_7)$ is the group $\{\pm I\}$.