18.701 Problem Set 10

1. Let V be the space of continuous complex-valued functions on the unit circle in the complex plane, and for $f, g \in V$, define

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(\theta)} g(\theta) d\theta.$$

(i) Show that this form is hermitian and positive definite.
(ii) Show that $T = i\frac{d}{d\theta}$ is a hermitian operator on V.
(iii) Let W be the subspace of V of functions $f(e^{i\theta})$, where f is a polynomial of degree $\leq n$. Find an orthonormal basis for W.

2. Chapter 7, Problem 7.4.

3. Let $\zeta = e^{2\pi i/n}$. The $n \times n$ *Fourier matrix* A is the complex symmetric matrix defined by $A_{ij} = \zeta^{ij}$. The row and column indices run from 0 to $n-1$. This matrix solves an interpolation problem, a discrete version of Fourier series: Given complex numbers $b_0, ..., b_{n-1}$, find a complex polynomial $f(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$ such that $f(\zeta^i) = b_i$.
(i) Explain how the matrix solves the problem.
(ii) An identity called *Fourier inversion* says that A is roughly its own inverse. Compute A^2 and A^*A, and find such an identity.
(iii) Show that A is a normal matrix, and determine its eigenvalues.

4. Let v be a fixed vector in $\mathbb{R}^3$, and let $\times$ denote the vector cross product that you learn in calculus. Let T be the linear operator $T(x) = (x \times v) \times v$.
(i) Show that this operator is symmetric. You may use properties of the scalar triple product, but not computation of the matrix.
(ii) Compute the matrix.

5. Let V denote $\mathbb{C}^n$ with its standard hermitian form $\langle X, Y \rangle = X^*Y$. Let A be a real orthogonal matrix.
(i) Let X be a complex eigenvector of A with complex eigenvalue $\lambda \neq \pm 1$. Show that the dot product X^tX is zero. (not the hermitian product X^*X.)
(ii) Writing $X = R + Si$, show that the real vector space spanned by the real vectors R, S is A-invariant, and describe the restriction of the operator A to this subspace.

6. Let $G = SL_2(\mathbb{R})$.
(i) Using conjugation by elementary matrices, show that every matrix $A \neq \pm I$ in G is conjugate to one of the form

$$\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & * \end{pmatrix}.$$

(ii) Show that the matrices in G with given trace are partitioned into at most three conjugacy classes.
(iii) Any element of G can be written in the form

$$A = \begin{pmatrix} r + x & y \\ z & r - x \end{pmatrix},$$

where $2r = \textit{trace}\, A$. Then the point (x, y, z) lies on the quadric $x^2 + yz = r^2 - 1$, Describe the quadrics that arise this way geometrically, and decompose them into conjugacy classes.