April 30, 2003

18.702 Problem Set 10

due Wednesday, May 14

Chapter 14, Problems 1.6, 3.4, 3.11, 6.26, 6.27a, 6.31c, 8.4, Misc. 1.

I. Determine the Galois groups of the splitting fields of the following polynomials over $\mathbb{Q}$:
(a) $x^4 - x^3 + x^2 - x + 1$,
(b) $x^4 + 2x^3 + x^2 + 3x + 2$,
(c) $x^4 + x + 1$,
(d) $x^4 + 3x^2 - 10$,
(e) $x^4 + 4x^2 + 2$,
(f) $x^4 - 12x^2 + 4$.

II. (This is about the Riemann Existence Theorem for quadratic extensions.) Let F be the field of rational functions $\mathbb{C}(x)$. Let α be an element which generates a quadratic extension K/F, and let $\beta = u + v\alpha$, with $u, v \in F$ and $v \neq 0$.

(a) Let $f(x, y) \in \mathbb{C}[x, y]$ and $g(x, z) \in \mathbb{C}[x, z]$ be the primitive irreducible polynomials such that $f(x, \alpha) = 0$ and $g(x, \beta) = 0$. Show how to determine f from g, being careful about denominators.

(b) Let the corresponding Riemann surfaces be $S : \{f = 0\}$ and $T : \{g = 0\}$. Show that the substitution $z = u + vy$ defines a map $S' \to T$, thereby showing that the Riemann surface is essentially independent of the choice of generator for K.

(c) Classify the quadratic extensions of F by deciding when two square root extensions are isomorphic. Explain how the quadratic extensions correspond to double coverings of the complex plane P.