

Relating a Riemann Surface to its Function Field

The Riemann surface of a polynomial contains the information needed to answer questions of the following type:

Question 1.1: Let F be the field of rational functions $\mathbb{C}(x)$, and let K be the extension field obtained by adjoining a root of $f(x, y) = y^3 + xy + x^2$, i.e., $K = F[y]/(f)$. Is the cube root of x in K ?

Because both $z^3 - x$ and f have degree 3, this is not immediately clear, and it is unpleasant to verify algebraically. However, let's call S the Riemann surface $\{f(x, y) = 0\}$ and T the Riemann surface $\{z^3 - x = 0\}$. Then S is branched at two points: $x = 0$ and $x = -4/27$, while T is branched only at $x = 0$. These facts are enough to conclude that the answer to the question is: *No*.

To see why, we have to make two things explicit: What it means for a second polynomial $g(x, z)$ to have a root in K , and how one Riemann surface is mapped to another.

I. *When does $g(z)$ have a root in $K = F[y]/(f)$?*

Let's start with any field F and an extension K of the form $K = F[y]/(f)$, where $f \in F[y]$ is an irreducible polynomial of degree n . Let α denote the residue of y in K , the root of f that we have adjoined. So $(1, \alpha, \dots, \alpha^{n-1})$ is an F -basis for K .

Let $g \in F[z]$ be another irreducible polynomial which happens to have a root β in K . (It seems advisable to introduce a new variable z .) Let L denote the field $F[z]/(g)$ obtained by adjoining a root of g to F formally. Let's call β_1 the root of g in L . Substitution of β for z defines a homomorphism $\eta : F[z] \rightarrow K$ which is the identity on F and which has g in its kernel. Since g is irreducible, the principal ideal (g) is maximal. Therefore $(g) = \ker \eta$, and by the First Isomorphism Theorem, $L = F[z]/(g)$ is isomorphic to its image in K , which is $F(\beta)$.

Corollary 1.2. *A root β of $g(z)$ in K defines an injective homomorphism $\eta : L \rightarrow K$ which is the identity on F . Conversely, such a homomorphism provides us with a root of g in K , namely $\eta(\beta_1)$.*

To interpret what it means for g to have a root in K in the polynomial algebra, we write the root β in terms of the basis of K , say $\beta = c_{n-1}\alpha^{n-1} + \dots + c_1\alpha + c_0$. Let

$$(1.3) \quad \tilde{p}(y) = c_{n-1}y^{n-1} + \dots + c_1y + c_0,$$

so that $\beta = \tilde{p}(\alpha)$ is the residue of $\tilde{p}(y)$ in K . The statement that β is a root of $g(z)$ means that the residue of $g(\tilde{p}(y))$ in K is zero, which means that $g(\tilde{p}(y))$ is divisible by $f(y)$. This shows

Proposition 1.4. *Let $f(y), g(z)$ be irreducible polynomials with coefficients in F , and let $K = F[y]/(f)$. Then g has a root in K if and only if there is a polynomial $\tilde{p}(y)$ of degree $< n = \deg f$, such that f divides $g(\tilde{p}(y))$.*

Unfortunately, deciding whether or not such a polynomial exists is a nonlinear problem which is usually hard to solve by direct computation. (Try it for the polynomials $f = y^3 + xy + x^2$ and $g = z^3 - x$.)

In our case, in which F is the field of rational functions $\mathbb{C}(x)$, we can translate this proposition as follows: We let f be a irreducible polynomial in $\mathbb{C}[x, y]$, and $K = F[y]/(f)$. Let $g = g(x, z)$ be another polynomial. We always suppose f and g aren't polynomials in x alone. Then by a root of g in K we mean an element β of K such that $g(x, \beta) = 0$ in K . The proposition tells us that such a root is determined by a polynomial $\tilde{p}(y) \in F[y]$ of degree $< \deg_y f$ such that $f(x, y)$ divides $g(x, \tilde{p}(y))$ in $F[y]$. The only point to notice is that the polynomial $\tilde{p}(y)$ has coefficients in F . It need not be in $\mathbb{C}[x, y]$. It is allowed to have a denominator, which must, however, be a polynomial in x alone. Let $d(x)$ be the

denominator, so that $d(x)\tilde{p}(y) = p(x, y)$, for some polynomial $p(x, y) \in \mathbb{C}[x, y]$. The denominator of $g(x, \tilde{p}(y))$ will be a power of $d(x)$. So for suitable k ,

$$(1.5) \quad d(x)^k g(x, \tilde{p}(y)) = h(x, y),$$

and $h(x, y) \in \mathbb{C}[x, y]$. We can clear this denominator, i.e., replace $g(x, \tilde{p}(y))$ by $h(x, y)$, and then testing divisibility by f in $\mathbb{C}[x, y]$ (see Ch 11, 3.9). But we aren't allowed to clear the denominator in $\tilde{p}(y)$, because that wouldn't preserve the property of representing a root.

II. Mapping one Riemann surface to another.

Let $f(x, y), g(x, z)$ be two irreducible polynomials and let $S : \{f = 0\}$ and $T : \{g = 0\}$ be their Riemann surfaces. Suppose that $g(x, z)$ has a root β in $K = F[y]/(f)$, and say that β is the residue of the polynomial $\tilde{p}(y) \in F[y]$, as above. Then as we saw, f divides $g(x, \tilde{p}(y))$ in $F[y]$ and f divides $h(x, y)$ in $\mathbb{C}[x, y]$.

We define a map

$$(2.1) \quad \Phi : \mathbb{C}_{x,y}^2 \rightarrow \mathbb{C}_{x,z}^2$$

by $\Phi(x, y) = (x, \tilde{p}(y))$. This map is defined at all points (x_0, y_0) except those for which x_0 is a root of the denominator $d(x)$, because if $d(x_0) \neq 0$, then substitution of (x_0, y_0) for (x, y) into $\tilde{p}(y)$ is permitted.

Proposition 2.2. *Let S' denote the complement of the finite set of points (x_0, y_0) of S such that x_0 is a root of $d(x)$. The restriction of Φ to S' defines a map $\phi : S' \rightarrow T$.*

Proof. Suppose that $(x_0, y_0) \in S'$, i.e., that $f(x_0, y_0) = 0$, but x_0 is not a root of $d(x)$. We must show that $\phi(x_0, y_0) \in T$. This is true because, since $f(x, y)$ divides $g(x, \tilde{p}(y))$ and $f(x_0, y_0) = 0$, $g(x_0, \tilde{p}(y_0)) = 0$ (see 1.5). \square

Corollary 2.3. *A root of $g(x, z)$ in K , or equivalently, a homomorphism $L \rightarrow K$ which is the identity on F , defines a map of Riemann surfaces $\phi : S' \rightarrow T$, defined on a subset $S' \subset S$ whose complement is finite, and which is compatible with the projections onto the x -plane P .*

In Question 1.1, the polynomials were $f = y^3 + xy + x^2$ and $g = z^3 - x$. In this case, because S is ramified at $x = -4/27$ but T is not, there is no map $S \rightarrow T$ compatible with the projections to P . So g does not have a root in K .

Example 2.4: Let $f(x, y) = y^2 - x$ and $g(x, z) = xz - 1$. Then S is the locus $\{y^2 - x\}$ in \mathbb{C}^2 , and T is the complex hyperbola $\{xz = 1\}$. Obviously g has the root $z = 1/x$ in K , and $L \approx F$. The map $L \rightarrow K$ is the inclusion of F into K . The map $S \rightarrow T$ sends (x, y) to $(x, 1/x)$, and is defined except at $x = 0$.

Riemann's theorem (Chapter 13, 7.14) asserts a converse to Corollary 2.3:

Riemann Existence Theorem 2.5. (i) *Let S and T be the Riemann surfaces of irreducible polynomials f and g in $\mathbb{C}[x, y]$, and let $K = F[y]/(f)$. If there is a subset $S' \subset S$ whose complement is finite and a continuous map $S' \rightarrow T$ compatible with the projections to P , then g has a root in K .*

(ii) *Give an arbitrary finite branched covering B of P , there is a polynomial $f(x, y)$ whose Riemann surface is homeomorphic to B outside of a finite set.*