18.781 14 Apr 2003

Thrm: Any periodic simple cont. fraction is quad. irrational number and converse also holds.

Proof:

Last time: proved $\Rightarrow$: $\theta = \langle \overline{a_0, \ldots, a_{n-1}} \rangle$ $\theta = \langle a_0, \ldots, a_{n-1}, \theta \rangle = \frac{\theta h_{n-1} + h_{n-2}}{\theta k_{n-1} + k_{n-2}}$

Other direction:

$\xi = \xi_0 = \frac{m_0 + \sqrt{d}}{q_0}$, $q_0 \mid (d - m_0^2)$ $d, m_0, q_0 \in \mathbb{Z}$, d not a square

$a_i = \lfloor \xi_i \rfloor$, $\xi_{i+1} = \frac{1}{\xi_i - a_i}$

In this case: $a_i = \lfloor \xi_i \rfloor$ $\xi_i = \frac{m_i + \sqrt{d}}{q_i}$, $m_{i+1} = a_i q_i - m_i$ $q_{i+1} = \frac{d - m_{i+1}^2}{q_i}$

$q_i \mid (d - m_i^2)$.

This ξ_i are the same as the previous ξ_i's

Let $\xi_i' = \frac{m_i - \sqrt{d}}{q_i}$

$\xi = \frac{\xi_n h_{n-1} + h_{n-2}}{\xi_n k_{n-1} + k_{n-2}}$

$\xi' = \frac{\xi_n' h_{n-1} + h_{n-2}}{\xi_n' k_{n-1} + k_{n-2}} \Rightarrow \frac{-k_{n-2}}{k_{n-1}} \left(\frac{\xi_0' - h_{n-2}/k_{n-2}}{\xi_0' - h_{n-1}/k_{n-1}} \right) = \xi_n'$

$\lim_{n \to \infty}$ ——— $= \lim \frac{-k_{n-2}}{k_{n-1}} \left(\frac{\xi_0' - \xi_0}{\xi_0' - \xi_0} \right) = \lim_{n\to\infty} \xi_n'$, so for $n > N$ (N suff. large), $\xi_n' < 0$.

$\xi_n - \xi_n' > 0$, $n > N$ $\xi_n - \xi_n' = \frac{2\sqrt{d}}{q_n}$, so q_n positive, n suff. large.

Also, $q_n q_{n+1} = d - m_{n+1}^2 < d$ so $q_n \le d$, and q_n can only take on finitely many values.

$m_{n+1}^2 < m_{n+1}^2 + q_n q_{n+1} = d$, $q_{n+1} = \frac{d - m_{n+1}^2}{q_n}$

so $|m_{n+1}| < \sqrt{d}$ $n > N$

and so q_n and m_n only take finitely many values.

so there exists $j < k$ $m_j = m_k$ $q_j = q_k$ $\xi_j = \xi_k$

then $\xi = \langle a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_k} \rangle$

Thrm. ξ is purely periodic (i.e. $\xi = \langle \overline{a_0, \ldots, a_{r-1}} \rangle$) iff $\xi > 1$ and $-1 < \xi' < 0$. (ξ quad. irrational)

Proof:

$\xi_{i+1} = \frac{1}{\xi_i - a_i} \Rightarrow \frac{1}{\xi_{i+1}'} = \xi_i' - a_i$

suppose $\xi > 1$, $-1 < \xi' < 0$.

since $\xi_0 > 1$, $a_i \ge 1$ for all i.

Claim: $-1 < \xi_{i+1}' < 0$.

Reason: if $\xi_i' < 0$, then $\xi_i' - a_i < -1$, so $-1 < \frac{1}{\xi_i' - a_i} < 0$

Also: $0 < -\frac{1}{\xi_{i+1}'} - a_i < 1$, since $\frac{-1}{\xi_i'} - a_i = -\xi_i'$ $\Rightarrow a_i = \lfloor \frac{-1}{\xi_{i+1}'} \rfloor$

Know that $\exists j, k$, $j < k$ s.t. $\xi_j = \xi_k$ (since we know it's periodic)

$a_{j-1} = \lfloor \frac{-1}{\xi_j'} \rfloor = \lfloor \frac{-1}{\xi_k'} \rfloor = a_{k-1} \Rightarrow \xi_{j-1} = \xi_{k-1}$

Thus $\xi_0 = \xi_{k-j}$, and it's purely periodic

Conversely, if $\xi = \langle \overline{a_0, \ldots, a_{n-1}} \rangle$

since $a_0 = a_n$, and $a_n \ge 1$, $a_0 \ge 1 \Rightarrow \xi > 1$, since $a_0 = \lfloor \xi \rfloor$

$\xi = \langle a_0, \ldots, a_{n-1}, \xi \rangle = \dfrac{\xi h_{n-1} + h_{n-2}}{\xi k_{n-1} + \xi k_{n-2}}$ so

ξ satisfies $f(x) = x^2 k_{n-1} + x(k_{n-2} - h_{n-1}) - h_{n-2} = 0$.

Want: $\xi' \in (-1,0)$ Need: $f(0) < 0$ $f(-1) > 0$, which shows that there is a root in the interval, and since the roots are ξ + ξ', and $\xi > 0$, $\xi' \in (-1,0)$

compute:

$f(0) = -h_{n-2} < 0$

$f(-1) = (k_{n-2} + h_{n-2})(a_{n-1} - 1) + k_{n-3} + h_{n-3}$ (using det.)

$\geq k_{n-3} + h_{n-3} > 0.$