18.781 25 Apr 2003

$\mathbb{P}^2(\mathbb{R})$: eq. classes of triples $[a:b:c]$ $a,b,c \in \mathbb{R}$ not all zero

$[a:b:c] \sim [\lambda a : \lambda b : \lambda c]$ $\lambda \in \mathbb{R} - \{0\}$

$\mathbb{R}^2 \subset \mathbb{P}^2(\mathbb{R})$

$(a,b) \longmapsto [a:b:1]$ $[a:b:c] \sim [\frac{a}{c} : \frac{b}{c} : 1]$, $c \neq 0$

a little extra: $\{[a,b,0]\}$

Define $\mathbb{P}^2(\mathbb{Q})$, $\mathbb{P}^2(\mathbb{C})$

Do same construction with $\mathbb{R}$ replaced by $\mathbb{Q}$ ($\mathbb{C}$).

$[a:b:c]$, $a,b,c \in \mathbb{Q}$ ($\mathbb{C}$) not all zero $\sim$

up to equivalence: $[a:b:c] \sim [\lambda a, \lambda b, \lambda c]$, $\lambda \in \mathbb{Q} - \{0\}$ ($\mathbb{C} - \{0\}$)

Now $\mathbb{P}^2(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{R})$.

since $[a':b':c'] \sim_{\mathbb{Q}} [a:b:c]$ same as $\sim_{\mathbb{R}}$ (for $a,b,c,a',b',c' \in \mathbb{Q}$)

- obviously, if $\sim_{\mathbb{Q}}$, $\exists \lambda \in \mathbb{Q}$, so $\lambda \in \mathbb{R}$
- if $[a:b:c] \sim_{\mathbb{R}} [a':b':c']$ $\exists \lambda \in \mathbb{R}$ s.t. $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$,

  so $\lambda = \frac{a'}{a}$ must be rational

Now, $\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \mathbb{R} \cup \text{pt}$

$\{[a:b:1]\} \cup \{[a:1:0]\} \cup \{[1:0:0]\}$

and similarly, $\mathbb{P}^2(\mathbb{Q}) = \mathbb{Q}^2 \cup \mathbb{Q} \cup \text{pt}$

ex. $x^4 - y^4 - 1 = 0$ $C = \{\text{solutions}\} \subset \mathbb{Q}^2, \mathbb{R}^2, \mathbb{C}^2$

Get subset $C \subset \mathbb{P}^2(\mathbb{R})$.

Put variable $z$: $x^4 - y^4 - z^4 = 0$.

A triple $(a,b,c)$ satisfies $a^4 - b^4 - c^4 = 0$

iff $(\lambda a, \lambda b, \lambda c)$ satisfies the eqn.

Thus we can talk about an equiv. class satisfies eqn.

$C \subset \mathbb{P}^2(\mathbb{R})$, defined to be eqv. classes that satisfy eqn.

$= \mathbb{R}^2 \cup \overbrace{\mathbb{R} \cup \text{pt}}^{\infty}$

$C \cap \mathbb{R}^2$: triples $[a:b:1]$ that satisfy, i.e. $a^4 - b^4 - 1 = 0$

$C \cap \infty$: triples $[a:b:0]$ s.t. $a^4 - b^4 = 0 \Rightarrow a^4 = b^4$ $[a:b:0] \sim [\frac{a}{b} : 1 : 0] = [\pm 1 : 1 : 0]$

All real sol'n in $\mathbb{R}^2$:

$[\sqrt[4]{y^4 - 1} : y : 1]$ $[-\sqrt[4]{y^4 - 1} : y : 1]$

Claim: limit pt as $y \to \infty$.

look in diff. coordinates:

$\{\text{eqv. classes of } [a:b:c],\ b \neq 0\} \subset \mathbb{P}^2(\mathbb{R})$

$\{[a:1:c]\} \leftrightarrow \mathbb{R}^2$

$[a:1:c]$ satisfies $a^4 - 1 - c^4 = 0$

$y \neq 0$. $[\sqrt[4]{y^4 + 1} : y : 1] \sim [\sqrt[4]{\frac{y^4 - 1}{y}} : 1 : \frac{1}{y}] \to [1:1:0]$

Curve: polynomial in two variables.

$f(x,y) = \sum a_{ij} x^i y^j$; $\quad C: f(x,y)$.

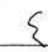
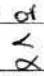
degree $= \max\{i+j\}$ $\quad d = \deg f(x,y)$.

Let $F(x,y,z) = \sum a_{ij} x^i y^j z^{d-i-j}$

$C(\mathbb{Q})$ = pts. in $\mathbb{P}^2(\mathbb{Q})$ which satisfy $F(x,y,z)=0$.

$C(\mathbb{R})$ = pts. in $\mathbb{P}^2(\mathbb{R})$; $\quad C(\mathbb{C})$ = pts. in $\mathbb{P}^2(\mathbb{C})$.

Next time: we'll define notion of smooth

Basic idea:  smooth  not smooth

Thm: (Faltings, 1983)

Start with $f(x,y) = \sum a_{ij} x^i y^j$, $a_{ij} \in \mathbb{Q}$, and $C$ (curve def. by $f(x,y)$) is smooth. Then if $\deg(f) \geq 4$, in $C(\mathbb{Q})$ finite.

ex: $x^d + y^d = z^d$. smooth (next time).

Up to mult. soln $(a,b,c)$ by $\lambda \in \mathbb{Q}-\{0\}$ there are only finitely many solns if $d \geq 4$.

i.e. $x^d + y^d - 1 = 0$ has only finitely many rational solns.

Look at rational sol'n of $f(x,y)$.

deg 1 or deg 2: totally understood; basically lines

deg $\geq 4$: finitely many sol'n

deg 3: elliptic curves