18.781: HOMEWORK SET 8

(0) Do the following problems from the book: 5.6: 2,5, 5.7: 3, 18

(1) Find the singular points of the curve $C \subset \mathbb{P}^2(\mathbb{C})$ defined by the equation

$$X^2 + Y^2 + X^2Y^2 = 0.$$

(2) Let $a, b \in \mathbb{Q}$. Show that the curve in $\mathbb{P}^2(\mathbb{C})$ defined by

$$Y^2 = X^3 + aX + b$$

has no singular points if $4a^3 + 27b^2$ is not zero.

(3) The purpose of this exercise is to solve the classical problem of finding all Pythagorean triples. That is, triples (a, b, c) which satisfy "Pythagoras equation"

$$a^2 + b^2 = c^2.$$

For each number t, let $L_t \subset \mathbb{P}^2(\mathbb{C})$ denote the line given by

$$\frac{1}{t}X + (Y - 1) = 0,$$

and let $C \subset \mathbb{P}^2(\mathbb{C})$ denote the curve given by the equation

$$X^2 + Y^2 = Z^2.$$

Show that for each t, the intersection $L_t \cap C$ consists of the two points

$$[0 : 1 : 1], \quad [2t : t^2 - 1 : t^2 + 1].$$

Show that the points of C are in bijection with $\mathbb{P}^1(\mathbb{C})$ with the bijection given by

$$[t : 1] \mapsto [2t : t^2 - 1 : t^2 + 1], \quad [1 : 0] \mapsto [0 : 1 : 1].$$

Deduce that the complex solutions to Pythagoras equation are

$$x = 2\lambda\mu, \quad y = \lambda^2 - \mu^2, \quad z = \lambda^2 + \mu^2$$

with $\lambda, \mu \in \mathbb{C}$.

Show that the real solutions are

$$x = 2\lambda\mu, \quad y = \lambda^2 - \mu^2, \quad z = \pm(\lambda^2 + \mu^2)$$

with $\lambda, \mu \in \mathbb{R}$.

Show that the integer solutions are

$$x = 2\lambda\mu\nu, \quad y = (\lambda^2 - \mu^2)\nu, \quad z = (\lambda^2 + \mu^2)\nu,$$

where λ and μ are coprime integers not both odd and $\nu \in \mathbb{Z}$, or

$$x = \lambda\mu\nu, \quad y = \frac{1}{2}(\lambda^2 - \mu^2)\nu, \quad z = \frac{1}{2}(\lambda^2 + \mu^2)\nu,$$

where λ and ν are coprime odd integers and $\nu \in \mathbb{Z}$.

Correction: The last line of the last problem should read "lambda and mu are coprime integers and nu is an integer". Some handwritten solutions to problem set 8 are hanging outside my door. Feel free to make a copy of them, but please don't steal them! (stealing is defined to be any activity which makes the solutions unavailable to other students).