18.781 14 May 2003

Office hours for final:

Martin: Th. May 15 9-12 am

May 23 9-12 am

Ian & Francesca: Wed May 21 (time & location on website)

What about the rest of $E(\mathbb{Q})$? (i.e. the not torsion part)

Thm: (Mordell-Weil) There exist unique integer $r \geq 0$ s.t.

$(E(\mathbb{Q}), +) \approx E(\mathbb{Q}) \oplus \mathbb{Z}^r$ $(\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ r times, i.e. vectors of length $r)$

r is called the rank.

Lemma: If G is an abelian group s.t. $\exists\, g_1, \ldots, g_r \in G$ s.t.

any elt. $g \in G$ can be written

$g = \sum n_i g_i$, $n_i \in \mathbb{Z}$

then $G \approx \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_s) \oplus \mathbb{Z}^r$

Mordell-Weil theorem $\Longrightarrow$ $\exists$ points $P_1, \ldots, P_n \in E(\mathbb{Q})$ s.t. any rat'l pt. $P \in E(\mathbb{Q})$

$\exists\, n_i$ s.t. $P = \sum n_i P_i$

The rank r is very mysterious:

- not known how big it can get

24 is known

- most examples have rank 0 or 1

Birch, Swinnerton-Dyer Conjecture:

L-series: $E: y^2 = x^3 + Ax + B$. $A, B \in \mathbb{Z}$

$E_p: y^2 \equiv x^3 + Ax + B \pmod p$

Talk about sol'n of E_p being smooth: not all $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ zero. (mod p).

From excercise: if $p \nmid 16(4A^3 + 27B^2)$ then every pt. is smooth.

$a_p = \#$ sol'n to E_p.

Define $L_p(E,T) = 1 - a_p T + pT^2$

if $p \mid 16(4A^3 + 27B^2)$

$L_p(E,T) = \begin{cases} 1-T \\ 1+T \\ 1 \end{cases}$ choose so that $L_p(E, 1/p) = \frac{\#\text{ of smooth pts of } E_p}{p}$

Define $L(E,s) = \prod_p \frac{1}{L_p(E, p^{-s})}$

It turns out $L(E,s)$ is a very nice function.

- Can rewrite using Taylor series around T

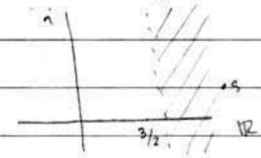
$L(E,s) = \prod_p (1 + c_{1p} p^{-s} + c_{2p} p^{-2s} + \dots)$

since $\frac{1}{1 \mp a_p T + p T^2} = 1 + c_{1p} T + c_{2p} T^2 + \dots$

Then $\prod_p (1 + c_{1p} p^{-s} + \dots) = \sum_n c_n n^{-s}$ (by prime factorization).

· Remarkable Thing # 1:

- This sum converges for $Re(s) > 3/2$



· In fact, Taniyama-Shimura conj $\Rightarrow L(E,s)$ extends to <u>analytic function</u> $L^*(E,s)$ on all of $\mathbb{C}$

· complex function f s.t. for any $s \in \mathbb{C}$, $\exists$ some $\varepsilon > 0$ s.t.

$f(z) = \sum a_n (z-s)^n$ for $|z-s| < \varepsilon$

Then, Birch, Swinnerton-Dyer conj:

At $s=1$, we can write

$L^*(E,z) = a_r (z-1)^r + a_{r+1} (z-1)^{r+1} + \dots, \ a_r = 0.$

Conjecture is that this r = rank of E.

So if $L^*(E,1) \neq 0 \Rightarrow$ rank of E is zero.

(this mean there's a constant term...)

Taniyama-Shimura $\Rightarrow L(E,s) = \sum_n c_n n^{-s}$

modular form: $g(z) = \sum_n c_n e^{2\pi i z n}$ (by some transform)

$g(z)$ is a fnc on $\mathfrak{h} = \{z \in \mathbb{C} \mid Im(z) > 0\}$