18.781 — 2 Apr 2003

Continued Fractions

algo habra sucedido o no y no me lo han dicho pero hoy he sido vendida

Notation: $x_0, x_1, \ldots, x_n$ real numbers, $x_1, \ldots, x_n > 0$

Define $\langle x_0, x_1, \ldots, x_n \rangle = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cdots \, x_{n-1} + \frac{1}{x_n}}}}$

e.g. $\langle x_0, x_1 \rangle = x_0 + \frac{1}{x_1}$

$\langle x_0, x_1, x_2 \rangle = x_0 + \cfrac{1}{x_1 + \frac{1}{x_2}}$

$\langle x_0, x_1, x_2, x_3 \rangle = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \frac{1}{x_3}}}$

In general, $\langle x_0, \ldots, x_n \rangle = x_0 + \frac{1}{\langle x_1, \ldots, x_n \rangle}$

Def. If x_i's are all integers, $\langle x_0, \ldots, x_n \rangle$ is called a simple continued fraction.

- Every rational number can be written as a simple continued fraction $\langle a_0, \ldots, a_j \rangle$

e.g. $\frac{3}{17} = 0 + \frac{1}{17/3} = 0 + \frac{1}{5 + 2/3} = 0 + \cfrac{1}{5 + \frac{1}{3/2}} = 0 + \cfrac{1}{5 + \cfrac{1}{1 + \frac{1}{2}}} = \langle 0, 5, 1, 2 \rangle$

In general, for a rational number u_0/u_1, $(u_0, u_1) = 1$

Euclidean Algorithm:

$$u_0 = u_1 a_0 + u_2 \qquad 0 < u_2 < u_1$$
$$u_1 = u_2 a_1 + u_3 \qquad 0 < u_3 < u_2$$
$$\vdots$$
$$u_{j-1} = u_j a_{j-1} + u_{j+1} \qquad 0 < u_{j+1} < u_j$$
$$u_j = u_{j+1} a_j \qquad \text{(note that since } (u_0, u_1) = 1,\ u_{j+1} = 1)$$

Let $\xi_i = u_i / u_{i+1}$

Divide 1st eqn by u_1:

$$\xi_0 = a_0 + \frac{1}{u_1/u_2} = a_0 + \frac{1}{\xi_1}$$
$$\xi_1 = a_1 + \frac{1}{\xi_2}$$
$$\vdots$$
$$\xi_{j-1} = a_{j-1} + \frac{1}{\xi_j}$$
$$\xi_j = a_j$$

$$\frac{u_0}{u_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, a_{j-1} + \frac{1}{a_j}}}}$$

e.g. $\frac{3}{17}$. $u_0 = 3 \quad u_1 = 17$

$3 = 0 \cdot 17 + 3$

$17 = 5 \cdot 3 + 2$

$3 = 1 \cdot 2 + 1$

$2 = 2 \cdot 1$

By algorithm, $\frac{3}{17} = \langle 0, 5, 1, 2 \rangle$, which agrees with previous result.

Expression of rat'l number as continued fraction is not unique.

$\frac{1}{1+\frac{1}{1}} = \frac{1}{2}$

$\langle a_0, a_1, \ldots, a_{j-1}, a_j \rangle = \langle a_0, a_1, \ldots, a_{j-1}, a_j - 1, 1 \rangle$, $a_j > 1$.

since $\frac{1}{a_{j-1} + \frac{1}{a_j}} = \frac{1}{a_{j-1} + \frac{1}{(a_j - 1) + \frac{1}{1}}}$

Lemma: If a_i's integers, then $\langle a_0, \ldots, a_n \rangle$ is a rational number.

Induction on number of a_i's

$n=0$, $\langle a_0 \rangle = a_0$. True.

$\langle a_0, \ldots, a_n \rangle = a_0 + \frac{1}{\langle a_1, \ldots, a_n \rangle}$, and if $\langle \ldots, \ldots \rangle$ is rational for n terms,

$\langle a_0, \ldots, a_n \rangle$ is rational (for $n+1$ terms)

Theorem: Say $\langle a_0, \ldots, a_j \rangle = \langle b_0, \ldots, b_n \rangle$, both simple, and $a_j > 1$, $b_n > 1$.

Then $j = n$ and $a_i = b_i$, $\forall i$.

Proof: Let $y_i = \langle b_i, \ldots, b_n \rangle$

$y_i = b_i + \frac{1}{\langle b_{i+1}, \ldots, b_n \rangle} = b_i + \frac{1}{y_{i+1}}$

$y_i > b_i$ for $i = 1, \ldots, n-1$, $y_i > 1$, $y_n = b_n > 1$

$b_i =$ greatest int. $\leq y_i$ for $i = 0, \ldots, n$.

since $0 < \frac{1}{y_{i+1}} < 1$.

Similarly, $a_i =$ greatest int. $\leq \langle a_i, \ldots, a_j \rangle$, all $i = 0, \ldots, j$

So $a_0 =$ greatest. int. $\leq \langle a_0, \ldots, a_j \rangle =$ greatest int. $\leq \langle b_0, \ldots, b_n \rangle = b_0$

Now, $\frac{1}{\langle a_1, \ldots, a_j \rangle} = \langle a_0, \ldots, a_j \rangle - a_0 = \langle b_1, \ldots, b_n \rangle - b_0 = \frac{1}{\langle b_1, \ldots, b_n \rangle}$

so $\langle a_1, \ldots, a_j \rangle = \langle b_1, \ldots, b_n \rangle$ since they're rational numbers.

By induction, $j = n$ and $a_i = b_i$ for $1 \leq i \leq n = j$

Summary: Any simple continued fraction represents a rational number, and any rational number can be expressed as a simple continued fraction in exactly two ways.