18.781 14 Feb 2003.

New Office Hours:

M 11-12 R 5-6 Rm 2-172

HW2: 1.3 36, 37 1.4 4, 8 2.1 5, 6, 10, 12

Primes

- there are infinitely many
- arbitrarily large intervals b/w consecutive primes } last time

consider function $\pi(x)$ = number of primes $\le x$.

Thrm (Prime number thrm):

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1$$

This proof is beyond the scope of this course ...sigh...

Thrm: $\sum_{\substack{p\le y \\ \text{primes}}} \frac{1}{p} > \log(\log y) - 1$

Proof: Can assume y integer.

$\mathcal{N} = \{n \mid \text{prime factors of } n \text{ are all } \le y\} \cup \{1\}$

$\sum_{n\in\mathcal{N}} \frac{1}{n} = \prod_{p\le y} (1 + \frac{1}{p} + \frac{1}{p^2} + \dots)$ since $\frac{1}{n} = \frac{1}{p_1^{\alpha_1}\cdots p_r^{\alpha_r}}$ where $p_i \le y$, by def $\mathcal{N}$

$\sum_{n\in\mathcal{N}} \frac{1}{n} > \sum_{n=1}^{y} \frac{1}{n}$, since prime factors of $n \le y$ are $\le y$.

$\sum_{n=1}^{y} \frac{1}{n} \ge \int_1^{y+1} \frac{dx}{x} = \log(y+1) > \log(y)$ by integral test, from calculus.

$\prod_{p\le y} (1 + \frac{1}{p} + \frac{1}{p^2} + \dots) = \prod_{p\le y} (1 - \frac{1}{p})^{-1}$ $(1 + \frac{1}{p} + \frac{1}{p^2} + \dots)(1 - \frac{1}{p}) = 1 + \frac{1}{p} + \frac{1}{p^2} + \dots - \frac{1}{p} - \frac{1}{p^2} - \dots$

Lemma: $\forall v$ $0 \le v \le \frac{1}{2}$, $e^{v+v^2} \ge \frac{1}{(1-v)}$. Proof later.

Given lemma, $\prod_{p\le y} e^{\frac{1}{p}+\frac{1}{p^2}} > \prod \frac{1}{(1-\frac{1}{p})} > \log(y)$, from above.

Take log both sides

$\Rightarrow \sum (\frac{1}{p} + \frac{1}{p^2}) > \log(\log y)$

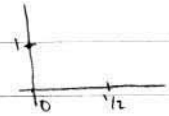
$\sum_{p\le y} \frac{1}{p} + \sum_{p\le y} \frac{1}{p^2} > \log(\log y)$

By integral test: $\sum_{p\le y} \frac{1}{p^2} < \sum_{n=2}^{\infty} < \int_1^{\infty} \frac{1}{x^2} dx = 1$

$\Rightarrow \sum_{p\le y} \frac{1}{p} > \log(\log y) - 1$

Proof of lemma:

Consider $f(x)=(1-x)e^{x+x^2}$. We want $f(x)\geq 1$, $0\leq x\leq \frac{1}{2}$.



If function increasing, we're done, since $f(0)=1$.

$f'(x) = -e^{x+x^2} + (1-x)(1+2x)e^{x+x^2} = x(1-2x)e^{x+x^2}$

If $0\leq x\leq \frac{1}{2}$, $x>0$, $1-2x>0$ & $e^{x+x^2}>0$ so

$f'(x)>0 \Rightarrow f(x)$ increasing.

Binomial Thrm

Def. Let $k\geq 0$ integer, a real number

Define binomial coeff. to be $\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}$

Note: If a integer, $\binom{a}{k} = \frac{a!}{k!(a-k)!}$ $(a\geq k)$

Thrm. If S set with a elements, $k\leq a$ integer, then

$\binom{a}{k}$ = number of subsets of S with k elts.

Proof: By induction on a.

If $a=1$, obviously.

General a, $S=S'\cup\{x_0\}$ where S' has $a-1$ elts.

The number of subsets of S with k elts =

(number of subsets of S' with k elts) + (number of subsets of S' with $k-1$ elts.)

since each subset of S is either contained in S' or contains x_0.

which, by induction, is $\binom{a-1}{k}+\binom{a-1}{k-1}$.

$\binom{a-1}{k}+\binom{a-1}{k-1} = \frac{(a-1)(a-2)\cdots(a-k)}{k!} + \frac{(a-1)(a-2)\cdots(a-k+1)}{(k-1)!} = \frac{(a-k+k)(a-1)(a-2)\cdots(a-k+1)}{k!} = \binom{a}{k}$