18.781 7 Apr 2003

$\langle a_0, a_1, \ldots \rangle$ infinite simple cont. fraction.

defined as irrational number $\xi = \lim_{n\to\infty} \underbrace{\langle a_0, \ldots, a_n \rangle}_{\text{convergents } = \frac{h_n}{k_n}}$

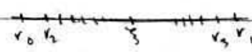
Thrm: Every irrational number can be written uniquely as an infinite simple continued fraction.

Proof:

Uniqueness:

$\langle a_0, a_1, \ldots \rangle = \langle b_0, b_1, \ldots \rangle$ then $a_i = b_i$ for all i.

Lemma: If $\xi = \langle a_0, a_1, \ldots \rangle$, then $a_0 = \lfloor \xi \rfloor$, where for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ = lrgest integer $n \le x$. (e.g. $\lfloor 7.4 \rfloor = 7$).

Proof: 

from last time.

$r_0 < \xi < r_1$. $r_0 = a_0$ (by def.) $r_1 = a_0 + \frac{1}{a_1}$, so since $a_1 \ge 1$, $r_1 = a_0 + \frac{1}{a_1} \le a_0 + 1$.

Thus $a_0 < \xi < a_0 + 1$, so $a_0 = \lfloor \xi \rfloor$.

Now, prove uniqueness by induction on i.

$i = 0$, $a_0 = \lfloor \langle a_0, a_1, \ldots \rangle \rfloor = \lfloor \langle b_0, \ldots \rangle \rfloor = b_0$

So assume true for ith term, prove for $i+1$th term

$\langle a_0, a_1, \ldots \rangle = \lim_{n\to\infty} \langle a_0, \ldots, a_n \rangle = \lim_{n\to\infty} \left(a_0 + \frac{1}{\langle a_1, \ldots, a_n \rangle} \right) = a_0 + \frac{1}{\lim_{n\to\infty} \langle a_1, \ldots, a_n \rangle} = a_0 + \frac{1}{\langle a_1, \ldots \rangle}$

Thus $\langle a_0, \ldots \rangle = a_0 + \frac{1}{\langle a_1, \ldots \rangle} = \langle b_0, b_1, \ldots \rangle = b_0 + \frac{1}{\langle b_1, \ldots \rangle}$

so since $a_0 = b_0$, $\langle a_1, \ldots \rangle = \langle b_1, \ldots \rangle$

Thus a_i, which was $i+1$th term, is ith term, so by induction, $a_i = b_i$.

Thus we've proved uniqueness.

Existence:

Start with irrational number ξ.

Guess: $a_0 = \lfloor \xi \rfloor$. $\xi = a_0 + \frac{1}{\xi_1}$ Then $\lfloor \xi_1 \rfloor = a_1$, ... etc.

Define: $a_i = \lfloor \xi_i \rfloor$, $\xi_{i+1} = \frac{1}{\xi_i - a_i}$.

To show: $\langle a_0, a_1, \ldots \rangle = \xi$.

Remark: $\xi_i > 1$, all i, and $\xi_i > a_i$

since $\xi_{i+1} = \frac{1}{\xi_i - a_i}$ is an irrational number by induction, and since $\xi_i - a_i \in (0,1)$, $\frac{1}{\xi_i - a_i} > 1$

Recall: $h_{-2} = 0$ $h_{-1} = 1$ $h_i = a_i h_{i-1} + h_{i-2}$

$k_{-2} = 1$ $k_{-1} = 0$ $k_i = a_i k_{i-1} + k_{i-2}$

$r_n = \langle a_0, \ldots, a_n \rangle = h_n / k_n$.

Also, by definition of ξ_i's,

$\xi = \langle a_0, \ldots, a_{n-1}, \xi_n \rangle$ (e.g. $\xi = a_0 + \frac{1}{\xi_1} = \langle a_0, \xi_1 \rangle$).

We showed last time that

$$\xi = \langle a_0, \ldots, a_{n-1}, \xi_n \rangle = \frac{\xi_n h_{n-1} + h_{n-2}}{\xi_n k_{n-1} + k_{n-2}},$$

$$\xi - \langle a_0, \ldots, a_{n-1} \rangle = \frac{\xi_n h_{n-1} + h_{n-2}}{\xi_n k_{n-1} + k_{n-2}} - \frac{h_{n-1}}{k_{n-1}} = \frac{(\xi_n h_{n-1} k_{n-1} + k_{n-1} h_{n-2}) - h_{n-1}(\xi_n k_{n-1} + k_{n-2})}{k_{n-1}(\xi_n k_{n-1} + k_{n-2})}$$

$$= \frac{-h_{n-1}k_{n-2} + k_{n-1}h_{n-2}}{k_{n-1}(\xi_n k_{n-1} + k_{n-2})} = \frac{(-1)^n}{k_{n-1}(\xi_n(k_{n-1}) + k_{n-2})}, \text{ is in abs. val} \le \frac{1}{k_{n-1}k_{n-2}}$$

Take $\lim_{n \to \infty} \frac{-(-1)^n}{k_{n-1}(\xi_n(k_{n-1}) + k_{n-2})} = 0$, since k_i increasing to infinity.

Ex: What is $\langle 1, 7, 1, 7, 1, 7, \ldots \rangle$?

Let $\xi = \langle 1, 7, 1, 7, 1, 7, \ldots \rangle$

$$\xi = 1 + \frac{1}{\langle 7, 1, 7, 1, 7, \ldots \rangle} = 1 + \frac{1}{7 + \frac{1}{\langle 1, 7, 1, 7, \ldots \rangle}} = 1 + \frac{1}{7 + \frac{1}{\xi}}$$

So $\xi = 1 + \frac{1}{7 + \frac{1}{\xi}} = 1 + \frac{\xi}{7\xi + 1} \Rightarrow \xi(7\xi + 1) = 7\xi + 1 + \xi \Rightarrow$

$7\xi^2 - 7\xi - 1 = 0 \Rightarrow \xi = \frac{7 \pm \sqrt{49 - 4 \cdot 7 \cdot -1}}{14} = \frac{7 \pm \sqrt{77}}{14} = \frac{1}{2} \pm \frac{\sqrt{77}}{14}$, so

$\xi = \frac{1}{2} + \frac{\sqrt{77}}{14}$, since we know $\xi > 0$.

Thm (Hurwitz):

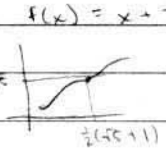
Given irrational ξ, there exists infinitely many rat'l numbers h/k with $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$

(False if you replace $\sqrt{5}$ by bigger number)

Proof:

Lemma: If $x > 1$, real an $x + \frac{1}{x} < \sqrt{5}$, then $x < \frac{1}{2}(\sqrt{5} + 1)$ and $\frac{1}{x} > \frac{1}{2}(\sqrt{5} - 1)$

Proof: $f(x) = x + \frac{1}{x}$ is increasing for $x > 1$ and $f(\frac{1}{2}(\sqrt{5} + 1)) = \sqrt{5}$



Sol: Write $\xi = \langle a_0, \ldots \rangle$. Every third $\frac{h_n}{k_n}$ satisfies inequality.

say ineq. fails for $\frac{h_{j-1}}{k_{j-1}}, \frac{h_j}{k_j}$ $q_j = k_j / k_{j-1}$.

$\left|\xi - \frac{h_{j-1}}{k_{j-1}}\right| + \left|\xi + \frac{h_j}{k_j}\right| \ge \frac{1}{\sqrt{5}k^2_{j-1}} + \frac{1}{\sqrt{5}k_j^2}$, with $\frac{h_{j-1}}{k_{j-1}}$ ξ $\frac{h_j}{k_j}$

so $\left|\frac{h_{j-1}}{k_{j-1}} - \frac{h_j}{k_j}\right| = \left|\xi - \frac{h_{j-1}}{k_{j-1}}\right| + \left|\xi + \frac{h_j}{k_j}\right|$

$= \left|\frac{h_{j-1}k_j - h_j k_{j-1}}{k_{j-1}k_j}\right| = \frac{1}{k_{j-1}k_j} \ge \frac{1}{\sqrt{5}k_{j-1}^2} + \frac{1}{\sqrt{5}k_j^2} \Rightarrow \sqrt{5} \ge \frac{k_j}{k_{j-1}} + \frac{k_{j-1}}{k_j} = q_j + \frac{1}{q_j}$

so $\frac{1}{2}(\sqrt{5} + 1) > q_{j+1} = a_{j+1} + q_j^{-1} > a_{j+1} + \frac{1}{2}(\sqrt{5} - 1)$.

if result fails for h_{j+1}/k_{j+1}, we get $\sqrt{5} >$