18.781 — 4 Apr 2003

$\langle a_0, \dots, a_n \rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \frac{1}{a_n}}}$

Say $a_0, a_1, \dots$ infinite seq. of integers $a_i > 0$ $i \geq 1$.
$\lim_{n\to\infty} \langle a_0, \dots, a_n \rangle$ exists and is an irrational number.

Define $\{h_n\}$ $\{k_n\}$.

$h_{-2} = 0$ $h_{-1} = 1$ $h_i = a_i h_{i-1} + h_{i-2}$ $h_0 = a_0$ $h_1 = a_1 a_0 + 1$

$k_{-2} = 1$ $k_{-1} = 0$ $k_i = a_i k_{i-1} + k_{i-2}$ $k_0 = 1$ $k_1 = a_1$

Prop. For every $x > 0$, $\langle a_0, \dots, a_{n-1}, x \rangle = \dfrac{x h_{n-1} + h_{n-2}}{x k_{n-1} + k_{n-2}}$

Proof: Induction on n.

$n = 1$. $\langle a_0, x \rangle = a_0 + \frac{1}{x}$

$\dfrac{x h_0 + h_{-1}}{x k_0 + k_{-1}} = \dfrac{x a_0 + 1}{x + 0} = a_0 + \frac{1}{x}$. ✓

General case:

$\langle \overbrace{a_0, \dots, a_n}^{n+1 \text{ terms}}, x \rangle = \langle \overbrace{a_0, \dots, a_n + \tfrac{1}{x}}^{n \text{ terms}} \rangle = \dfrac{(a_n + \frac{1}{x}) h_{n-1} + h_{n-2}}{(a_n + \frac{1}{x}) k_{n-1} + k_{n-2}} = \dfrac{x(a_n h_{n-1} + h_{n-2}) + h_{n-1}}{x(a_n k_{n-1} + k_{n-2}) + k_{n-1}} = \dfrac{x h_n + h_{n-1}}{x k_n + k_{n-1}}$. □

Def. Let $r_n = \langle a_0, \dots, a_n \rangle$

Cor: $r_n = \dfrac{h_n}{k_n}$

Proof: $r_n = \langle a_0, \dots, a_n \rangle = \dfrac{a_n h_{n-1} + h_{n-2}}{a_n k_{n-1} + k_{n-2}} = \dfrac{h_n}{k_n}$.

Thm:

i) $i \geq 1$ $h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}$ $\left(\Rightarrow r_i - r_{i-1} = \dfrac{(-1)^{i-1}}{k_i k_{i-1}} \quad (h_i, k_i) = 1 \right)$

ii) $i > 1$, $h_i k_{i-2} - h_{i-2} k_i = (-1)^i a_i$ $\left(\Rightarrow r_i - r_{i-2} = \dfrac{(-1)^i a_i}{k_i k_{i-2}} \right)$

Proof.

i) Induction on i

$i = 1$. $h_1 k_0 - h_0 k_1 = (a_0 a_1 + 1) - a_0 a_1 = 1 = (-1)^{1-1}$ ✓

General case:

$h_i k_{i-1} - h_{i-1} k_i = (a_i h_{i-1} + h_{i-2}) k_{i-1} - h_{i-1}(a_i k_{i-1} + k_{i-2}) = -(h_{i-1} k_{i-2} - h_{i-2} k_{i-1}) = -(-1)^{i-2} = (-1)^{i-1}$.

ii) $h_i k_{i-2} - h_{i-2} k_i = (a_i h_{i-1} + h_{i-2}) k_{i-2} - h_{i-2}(a_i k_{i-1} + k_{i-2}) = a_i h_{i-1} k_{i-2} - a_i h_{i-2} k_{i-1} = a_i (-1)^{i-2} = a_i (-1)^i$.

Cor: $r_0 < r_2 < r_4 < \dots < \dots < r_5 < r_3 < r_1$

$\lim_{n\to\infty} r_n$ exists, and $r_{2j} < \lim r_n < r_{2j+1}$

Proof: (ii) $\Rightarrow r_{2j} < r_{2j+2}$, $r_{2j-1} > r_{2j+1}$ since $\frac{a_i}{k_i k_{i-1}}$

(i) $\Rightarrow r_{2j} < r_{2j-1}$

$r_{2n} < r_{2n+2j} < r_{2n+2j-1} < r \leq r_{2j-1}$

Now $\{r_0, r_2, r_4, \ldots\}$ converges (increasing + bounded) so call r^- the limit.

Similarly, $\{r_1, r_3, r_5, \ldots\}$ converges to some r^+.

Want $r^- = r^+$

Reason: $|r_{2n} - r_{2n-1}| \to 0$, which follows from (i) since k_i strictly increasing positive sequence of integers.

$\lim_{i \to \infty} k_i = \infty$.

Define $\langle a_0, a_1, \ldots \rangle = \lim_{n \to \infty} \langle a_0, \ldots, a_n \rangle$.

<u>Thm</u>: $\langle a_0, a_1, \ldots \rangle$ irrational.

Proof: let $r = \langle a_0, a_1, \ldots \rangle$

$0 < |r - r_n| < |r_{n+1} - r_n| = \frac{1}{k_n k_{n+1}}$

Multiply by k_n: $0 < |k_n r - h_n| < \frac{1}{k_{n+1}}$.

Say $r = \frac{a}{b}$. Multiply by b: $0 < |k_n a - h_n b| < \frac{b}{k_{n+1}}$

For all n, $|k_n a - h_n b| \geq 1$ (integers), but $\lim_{n \to \infty} \frac{b}{k_{n+1}} = 0$.

contradiction, so r irrational.