18.781 9 Apr. 2003

$\langle a_0, a_1, \ldots \rangle$

$h_i = a_i h_{i-1} + h_{i-2}$

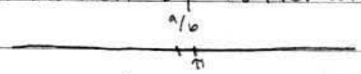
$k_i = a_i k_{i-1} + k_{i-2}$

$h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}$

ξ irrational. Write uniquely as

$\xi_0 = \langle a_0, a_1, \ldots \rangle \quad a_i = \lfloor \xi_i \rfloor \quad \xi_{i+1} = \frac{1}{\xi_i - a_i} \quad \lfloor \xi_0 \rfloor = a_0$

Approximation by rationals:



closer approximation involve increasing denom.

Idea: best you can do is $\frac{h_n}{k_n}$

Thrm: $\left|\xi - \frac{h_n}{k_n}\right| < \frac{1}{k_n k_{n+1}}$ or $|\xi k_n - h_n| < \frac{1}{k_{n+1}}$

Pf: $\left|\xi - \frac{h_n}{k_n}\right| = \frac{1}{k_n(\xi_{n+1} k_n + k_{n-1})} < \frac{1}{k_n(a_{n+1} k_n + k_{n-1})} = \frac{1}{k_n k_{n+1}}$

last time $\qquad a_{n+1} = \lfloor \xi_{n+1} \rfloor$

Thrm: For all n: $\left|\xi - \frac{h_n}{k_n}\right| < \left|\xi - \frac{h_{n-1}}{k_{n-1}}\right|$

stronger: $|\xi k_n - h_n| < |\xi k_{n-1} - h_{n-1}|$

$\Rightarrow \left|\xi - \frac{h_n}{k_n}\right| < \frac{1}{k_n}|\xi k_{n-1} - h_{n-1}| < \frac{1}{k_{n-1}}|\xi k_{n-1} - h_{n-1}| = \left|\xi - \frac{h_{n-1}}{k_{n-1}}\right|$

Proof: (Of stronger statement)

$\xi_n k_{n-1} + k_{n-2} < (a_n + 1)k_{n-1} + k_{n-2} = a_n k_{n-1} + k_{n-2} + k_{n-1} = k_n + k_{n-1} \leq a_{n+1} k_n + k_{n-1} = k_{n+1}$

$\left|\xi - \frac{h_{n-1}}{k_{n-1}}\right| = \frac{1}{k_{n-1}(\xi_n k_{n-1} + k_{n-2})} > \frac{1}{k_{n-1} k_{n+1}}$

$|\xi k_{n-1} - h_{n-1}| > \frac{1}{k_{n+1}} > |\xi k_n - h_n|$, since $k_{n+1} < \frac{1}{|\xi k_n - h_n|}$

Thrm: $\frac{a}{b} \in \mathbb{Q}$, $b > 0$. If $\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right|$ some n, then $b > k_n$.

In fact, if $|\xi b - a| < |\xi k_n - h_n|$ then $b \geq k_{n+1}$

Proof:

Second statement $\Rightarrow$ first

Say $\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right|$ but $b \leq k_n$

Then $|\xi b - a| < b\left|\xi - \frac{h_n}{k_n}\right| \leq k_n \left|\xi - \frac{h_n}{k_n}\right| = |\xi k_n - h_n|$

If second true, then $b \geq k_{n+1} > k_n$, so second false if first false.

Proof of second:

By contradiction.

$|\xi b - a| < |\xi k_n - h_n| \quad b < k_{n+1}$

Consider $\quad x k_n + y k_{n+1} = b$

$\qquad\qquad x h_n + y h_{n+1} = a$

Claim: can solve with x, y integers.

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} k_n & k_{n+1} \\ h_n & h_{n+1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} b \\ a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det} \begin{bmatrix} h_{n+1} & -k_{n+1} \\ -h_n & k_n \end{bmatrix} \quad \det = k_n h_{n+1} - k_{n+1} h_n = \pm 1.$$

Claim: neither x nor y is zero

$x = 0 \Rightarrow y k_{n+1} = b$, y int $\Rightarrow k_{n+1} \leq b$, by assumption can't happen

$y = 0 \Rightarrow x k_n = b \quad x h_n = a$

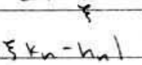
$|\xi b - a| = |\xi x k_n - x h_n| \geq |\xi k_n - h_n|$

Claim: x and y have opposite sign

$y < 0$. $x k_n = b - y k_{n+1} > 0$, since $k_n > 0$, $x > 0$

$y > 0$. $x k_n = b - y k_{n+1}$, since $b < k_{n+1}$ $b - y k_{n+1} < 0$, so $x < 0$

$|\xi b - a| = |x(\xi k_n - h_n) + y(\xi k_{n+1} - h_{n+1})|$

$\xi k_n - h_n$ and $\xi k_{n+1} - h_{n+1}$ have opposite sign 

$= |x(\xi k_n - h_n)| + |y(\xi k_{n+1} - h_{n+1})| > |x(\xi k_n - h_n)| \geq |\xi k_n - h_n|$

contradiction.

Thrm: ξ irrational, and $\frac{a}{b} \in \mathbb{Q}$, $b \geq 1$.

$|\xi - \frac{a}{b}| < \frac{1}{2b^2}$

Then $\frac{a}{b} = \frac{h_n}{k_n}$ for some n.

Proof: choose n s.t. $k_n \leq b < k_{n+1}$ (since $\{k_n\}$ increasing)

$|\xi b - a| < |\xi k_n - h_n|$ can't happen.

so $|\xi b - a| \geq |\xi k_n - h_n|$.

$|\xi k_n - h_n| \leq |\xi b - a| < \frac{1}{2b}$.

$|\xi - \frac{h_n}{k_n}| < \frac{1}{2bk_n}$

if $\frac{a}{b} \neq \frac{h_n}{k_n}$ $\quad \frac{1}{bk_n} < \frac{|bh_n - ak_n|}{bk_n} = |\frac{h_n}{k_n} - \frac{a}{b}| \leq |\xi - \frac{h_n}{k_n}| + |\xi - \frac{a}{b}| < \frac{1}{2bk_n} + \frac{1}{2b^2}$

$\frac{1}{bk_n} < \frac{1}{2bk_n} + \frac{1}{2b^2} \leq \frac{1}{2bk_n} + \frac{1}{2bk_n} = \frac{1}{bk_n}$. contradiction.

Thus. $\frac{a}{b} = \frac{h_n}{k_n}$