18.950 Handout 6. Some facts about harmonic functions.

Definition: Let $\Omega \subseteq \mathbf{R}^2$ be open. A C^2 function $u : \Omega \to \mathbf{R}$ is said to be *harmonic* in Ω if, in Ω,

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Theorem 1 (Mean Value Properties). *Suppose u is harmonic in Ω. Then u satisfies the following mean value properties. If $B_r(\mathbf{x}_0)$ is any ball with $\overline{B}_r(\mathbf{x}_0) \subseteq \Omega$ then*

(a) $u(\mathbf{x}_0) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{x}_0)} u\, ds$ *and*

(b) $u(\mathbf{x}_0) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{x}_0)} u(x,y)\, dxdy.$

Remarks: (1) The integral on the right hand side of (a) is a line integral with respect to arc length. Thus if $\mathbf{x}_0 = (a,b)$ then $\int_{\partial B_r(\mathbf{x}_0)} u\, ds \equiv \int_0^{2\pi} u(a + r\cos t, b + r\sin t) r\, dt$.

(2) The theorem says that the value of a harmonic function at a point is equal to both the average value of the function over a circle around that point (as in part (a)) and also to the average value of the function over a disk centered at that point (as in part (b)).

Proof. Since $\Delta\, u = \operatorname{div}(Du)$, we have by Green's theorem that

$$\int_{B_\rho(\mathbf{x}_0)} \Delta u dxdy = \int_{\partial\, B_\rho(\mathbf{x}_0)} -u_y dx + u_x dy$$

and hence, since $\Delta u = 0$,

$$\begin{aligned} 0 &= \int_{\partial\, B_\rho(\mathbf{x}_0)} -u_y dx + u_x dy \\ &= \rho \int_0^{2\pi} u_y(a + \rho\cos t, b + \rho\sin t)\sin t + u_x(a + \rho\cos t, b + \rho\sin t)\cos t\, dt \\ &= \int_0^{2\pi} \frac{d}{d\rho} u(a + \rho\cos t, b + \rho\sin t)\, dt \\ &= \frac{d}{d\rho} \int_0^{2\pi} u(a + \rho\cos t, b + \rho\sin t)\, dt. \end{aligned}$$

Integrating this with respect to ρ from 0 to r, we have that

$$\int_0^{2\pi} u(a + r\cos t, b + r\sin t)dt = 2\pi u(\mathbf{x}_0)$$

or, equivalently,

$$u(\mathbf{x}_0) = \frac{1}{2\pi r}\int_{\partial B_r(\mathbf{x}_0)} u\,ds.$$

This is part (a). To prove part (b), note that by changing variables to polar coordinates we have for any continuous function f that

$$\int_{B_r(\mathbf{x}_0)} f(x,y)dxdy = \int_0^r \int_0^{2\pi} f(a + \rho\cos t, b + \rho\sin t)\rho dt d\rho$$

or, equivalently, that

$$\int_0^r \int_{\partial B_\rho(\mathbf{x}_0)} f\,ds d\rho = \int_{B_r(\mathbf{x}_0)} f(x,y)dxdy. \tag{1}$$

Now replace r with ρ in the identity of part (a) and multiply the resulting identity by $2\pi\rho$ and integrate both sides with respect to ρ over $[0,r]$. In view of (1), this gives exactly the identity of part (b). □

Lemma 1 (Gradient estimate for non-negative harmonic functions). *Suppose u is non-negative and harmonic in Ω. If $\overline{B}_r(\mathbf{x}_0) \subseteq \Omega$, then*

$$|Du(\mathbf{x}_0)| \leq \frac{2\sqrt{2}}{r}u(\mathbf{x}_0).$$

Proof. Note first that if u is harmonic, then each of its partial derivatives u_x and u_y is also harmonic. This follows directly by differentiating the equation $\Delta u = 0$. Hence by the mean value identity (b) of Theorem 1, we have that

$$u_x(\mathbf{x}_0) = \frac{1}{\pi r^2}\int_{B_r(\mathbf{x}_0)} u_x(x,y)dxdy. \tag{2}$$

By Green's theorem

$$\int_{B_r(\mathbf{x}_0)} u_x dxdy = \int_{\partial B_r(\mathbf{x}_0)} u dy = \int_0^{2\pi} u(a + r\cos t, b + r\sin t)r\cos t dt.$$

Using this in (2) and keeping in mind that u is non-negative, we get

$$|u_x(\mathbf{x}_0)| \leq \frac{1}{\pi r^2} \int_0^{2\pi} u(a + r\cos t, b + r\sin t) r dt = \frac{1}{\pi r^2} \int_{\partial B_r(\mathbf{x}_0)} u\, ds.$$

Now use the mean value property (a) of Theorem 1 to deduce from the above that

$$|u_x(\mathbf{x}_0)| \leq \frac{2}{r} u(\mathbf{x}_0).$$

Similarly, we also have

$$|u_y(\mathbf{x}_0)| \leq \frac{2}{r} u(\mathbf{x}_0)$$

and combining these two estimates, we conclude that

$$|Du(\mathbf{x}_0)| \leq \frac{2\sqrt{2}}{r} u(\mathbf{x}_0).$$

□

Corollary 1. *If a harmonic function on the entire plane* $\mathbf{R}^2$ *is bounded above or below, then it must be a constant.*

Proof. Suppose u is harmonic everywhere in $\mathbf{R}^2$ and that $u(\mathbf{x}) \leq M$ for some constant M and all $\mathbf{x} \in \mathbf{R}^2$. Then $v(\mathbf{x}) = M - u(\mathbf{x})$ is harmonic in $\mathbf{R}^2$ and non-negative. Thus for *any* point $\mathbf{x} \in \mathbf{R}^2$, we have by Lemma 1 (with v in place of u and $\mathbf{x}$ in place of $\mathbf{x}_0$) that

$$|Du(\mathbf{x})| \leq \frac{2\sqrt{2}}{r}(M - u(\mathbf{x}))$$

for *all* $r > 0$. Letting $r \to \infty$, we conclude that $Du(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbf{R}^2$, and hence that u is a constant.

A similar argument applies to the case when u is bounded below. □