Chapter 5

Riemannian curvature

Spacetime curvature displays features similar to curvature of the sphere, as illustrated by parallel transport on the sphere in the previous chapter. The four-dimensional generalization is described by the Riemann tensor. The Riemann tensor and the metric in different ways describe the Newtonian limit as well as gravitational radiation.

5.1 Derivations of the Riemann tensor

The Riemann tensor has various representations which bring about various aspects of spacetime, arising from different derivations.

(a) Riemann tensor from loop integration.

Extending the discussion of parallel transport on the sphere, consider parallel transport of vectors along closed curves in spacetime. A vector a said to be parallelly transported along a curve with tangent u^a if

$$u^a\nabla_a\xi^b = u^a\left(\partial_a\xi^b + \Gamma^b_{ac}\xi^c\right) = 0, \tag{5.1}$$

where

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}\left(g_{eb,a} + g_{ae,b} - g_{ab,c}\right) \tag{5.2}$$

denotes the Christoffel connection in coordinate form.

Consider parallel transport of ξ^b along a closed loop $\gamma : x^b(s)$.[1] This introduces a descripancy between the initial state and final state of the vector, given by

$$\delta\xi^b = \xi^b_f - \xi^b_i = \int_\gamma d\xi^b = \int_\gamma u^a \partial_a \xi^b ds = -\int_\gamma u^a \Gamma^b_{ac}\xi^c ds, \tag{5.3}$$

where $u^a = dx^a/ds$. The leading order contribution to the integral derives from the linear variations in the integrand. Upon taking a Taylor series expansion in case of small loops in the neighborhood of the origin, we write

$$\left(\Gamma^b_{ac} + \partial_e \Gamma^b_{ac} x^e\right) u^a \left(\xi^c + \partial_e \xi^c x^e\right). \tag{5.4}$$

Terms linear in x^e are

$$u^a x^e \left(\partial_e \Gamma^b_{ac}\xi^c + \Gamma^b_{ac}\partial_e \xi^c\right), \tag{5.5}$$

where the factor in parenthesis is constant, evaluated at the origin. By $\int_\gamma u^a x^e ds = -\int_\gamma u^e x^a ds$ and $u^e\partial_e\xi^c = -u^e\Gamma^c_{ef}\xi^f$, we have

$$\delta\xi^b = -\left(\int_\gamma u^a x^e ds\right)\left(\partial_e\Gamma^b_{af} - \Gamma^b_{ac}\Gamma^c_{ef}\right)\xi^f = \frac{1}{2}\left(\int_\gamma u^e x^a ds\right) R^b{}_{fea}\xi^f. \tag{5.6}$$

Here, we define the Riemann tensor

$$R^b{}_{fea} = \partial_e \Gamma^b_{af} - \partial_a \Gamma^b_{ef} + \Gamma^b_{ce}\Gamma^c_{af} - \Gamma^b_{ca}\Gamma^c_{ef}. \tag{5.7}$$

By construction, the Riemann tensor is antisymmetric in its last two indices.

(b) Riemann tensor from non-cummutativity of covariant differentiation. Antisymmetric covariant derivatives reduce to a linear expression in the tensor at hand, namely

$$\nabla_a\nabla_b\xi_c - \nabla_b\nabla_a\xi_c = R_{abc}{}^d\xi_d. \tag{5.8}$$

Indeed, by explicit calculation

$$\nabla_{[a}\nabla_{b]}\xi_c = \nabla_{[a}\left(\partial_{b]}\xi_c - \Gamma^e_{b]c}\xi_e\right). \tag{5.9}$$

[1] See, e.g., 't Hooft, 2002, *Introduction to General Relativity* (Rindler Press, New Jersey); Landau & Lifschitz, 1984, *Classical Theory of Fields* (Pergamon 4th Ed).

The right hand-side expands into

$$\begin{array}{rl}\nabla_{[a}\nabla_{b]}\xi_c = & \partial_{[a}\partial_{b]}\xi_c - \partial_{[a}\Gamma^e_{b]c}\xi_e - \Gamma^e_{c[b}\partial_{a]}\xi_e \\ & -\Gamma^f_{[ab]}\left(\partial_f\xi_c - \Gamma^e_{fc}\xi_e\right) - \Gamma^f_{c[a}\left(\partial_{b]}\xi_f - \Gamma^e_{b]f}\xi_e\right)\end{array} \tag{5.10}$$

i.e.,

$$\nabla_{[a}\nabla_{b]}\xi_c = \left(\partial_{[b}\Gamma^e_{a]c} + \Gamma^f_{c[a}\Gamma^e_{b]f}\right)\xi_e. \tag{5.11}$$

This introduces (5.8).

5.2 Symmetries of the Riemann tensor

Symmetries of the Riemann tensor can be seen by inspection in a locally geodesic coordinate system. Thus, consider $\Gamma^c_{ab} \equiv 0$ and $\partial_c g_{ab} = 0$ at a point. We then have

$$R_{bfea} = g_{bc}R^c{}_{fea} = g_{bc}\left(\partial_e\Gamma^c_{af} - \partial_a\Gamma^c_{ef}\right). \tag{5.12}$$

Upon expansion, this gives

$$R_{bfea} = \partial_e\left(g_{bc}\Gamma^c_{af}\right) - \partial_a\left(g_{ac}\Gamma^c_{ef}\right) = \frac{1}{2}\left(g_{ab,fe} + g_{ef,ba} - g_{eb,fa} - g_{af,be}\right), \tag{5.13}$$

i.e.,

$$R_{bfea} = \frac{1}{2}\left(g_{ba,fe} + g_{fe,ab} - g_{be,af} - g_{af,eb}\right). \tag{5.14}$$

By inspection, we draw two conclusions

$$R_{bfea} = -R_{fbea} = -R_{bfae} = R_{fbae} = R_{eabf} \tag{5.15}$$

$$R_{bfea} + R_{beaf} + R_{bafe} = 0. \tag{5.16}$$

The first (5.15) shows that R_{bfea} is represented by a symmetric 6×6 matrix, which has 21 independent components. The second (5.16) is independent of the first (5.15) only for $bfea = 0123$ (or any permutation thereof), so that combined, the Riemann tensor has 20 independent components (and $N^2(N^2-1)/12$ independent components in $N-$dimensional spaces.)

Working in the same locally flat coordinate system, consider the derivative

$$\nabla_d R_{bfea} = \partial_d \left(g_{bc} \left(\partial_e \Gamma^c_{ef} - \partial_a \Gamma^c_{ef} \right) \right), \tag{5.17}$$

i.e.,

$$R_{bfea,d} = g_{bc} \left(\Gamma^c_{fa,de} - \Gamma^c_{fe,ad} \right). \tag{5.18}$$

This obtains the Bianchi indentity

$$\nabla_{[e} R_{ab]cd} = 0, \tag{5.19}$$

which holds covariantly following general coordinate transformations.

The contraction $R_{ac} = R_{abc}{}^b$ defines the Ricci tensor, and $R = R^c{}_c$ the scalar curvature. The Bianchi identify (5.19) defines the identity

$$\nabla^a G_{ab} \equiv 0 \tag{5.20}$$

for the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R. \tag{5.21}$$

This second form (5.20) of the Bianchi identity gives rise to the Einstein equations

$$G_{ab} = 8\pi T_{ab} \tag{5.22}$$

in the presence of a stress-energy tensor T_{ab}, satisfying conservation of energy and momentum,

$$\nabla_a T^{ab} = 0. \tag{5.23}$$

Since (5.22) is a covariant expression, (5.20) implies that (5.22) does *not* impose conditions on the second time-derivatives on the four functions g_{0a}. The g_{0a} are are not dynamical variables and represent freely specifiable functions: gauge functions which define slicing of spacetime in three-dimensional hypersurfaces.

5.3 Foliation in spacelike hypersurfaces

A time-coordinate t (with derivative vector $(\partial_t)^b$) and its hypersurfaces Σ_t of constant time come with two vectors:

$$\mathcal{N}_a = g_{ta}, \quad n_a = \partial_a t/\sqrt{-\partial_a t \partial^a t}, \tag{5.24}$$

where n_a denotes the unit normal ($n^2 = -1$) to Σ_t. (The vector $\mathcal{N}_a$ is commonly denoted by t_a, see, e.g., Wald (1984).) Generally, the covariant vectors $\mathcal{N}_a$ and n_a are independent. Marching from one hypersurface to the next brings along a variation dt, along with the covariant displacement

$$ds_a = \mathcal{N}_a dt. \tag{5.25}$$

The displacement ds_a expresses $\mathcal{N}_a$ as a "flow of time." It can be expressed in terms of orthogonal projections along n_a onto Σ_t in terms of the lapse function N and shift functions N_a,

$$\mathcal{N}^a = Nn^a + N^a. \tag{5.26}$$

Here $N = -\mathcal{N}_a n^a$ and $N_a = h_a^b \mathcal{N}_b$, expressed in the metric

$$h_{ab} = g_{ab} + n_a n_b \tag{5.27}$$

as the orthogonal projection of g_{ab} onto Σ_t. Note that $ds^2 = \mathcal{N}^2 dt^2 = g_{tt} dt^2$ as the square of (5.25), so that $g_{tt} = -N^2 + N_c N^c$. With $n_a = (n_t, 0, 0, 0)$, it follows that

$$g_{ab} = \begin{pmatrix} N^c N_c - N^2 & N_j \\ N_i & h_{ij} \end{pmatrix}, \tag{5.28}$$

where i, j refer to the spatial coordinates x^i of (t, x^i). The lapse function satisfies $\sqrt{-g} = N\sqrt{h}$. The four degrees of freedom in the five functions (N, N_a) are algebraically equivalent to $\mathcal{N}_a$. An equivalent expression for the line-element, in so-called 3+1 form [6], is

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{5.29}$$

where $\alpha = N$ is referred to as the redshift factor and $\gamma_{ij}\beta^j = g_{it}$.

The line-element (5.29) is instructive. It contains the previous Schwarzschild line-element with $\beta^j = 0$, that of a non-geodestic observer with $\beta_i = -\epsilon_{ijk} x^j \omega^k$ and, as will be seen later, the frame-dragging angular velocity β^ϕ around a Kerr black hole.

5.4 The Riemann tensor in connection form

Using the volume element $\epsilon_{abcd} = \Delta_{abcd}\sqrt{-g}$, where Δ_{abcd} denotes the totally antisymmetric symbol, we define the dual $*R_{abcd} = (1/2)\epsilon_{ab}{}^{ef}R_{efcd}$. The Bianchi identity then takes the form

$$\nabla^a * R_{abcd} = 0. \tag{5.30}$$

The Bianchi identity also gives $\nabla^d R_{abcd} = 2\nabla_{[b}R_{a]c}$. Combined with Einstein equations (5.22) this becomes

$$\nabla^a R_{abcd} = 16\pi\tau_{bcd}. \tag{5.31}$$

Here, we introduce

$$\tau_{bcd} = \left(\nabla_{[c}T_{d]b} - \frac{1}{2}g_{b[d}\nabla_{c]}T\right) \tag{5.32}$$

with T_c^c denoting the trace of the stress-energy tensor. This source term is divergence free:

$$\nabla^b\tau_{bcd} \equiv 0. \tag{5.33}$$

The equations (5.30) and (5.31) are in many ways analogous to Maxwell's equations. This can be made more explicit as follows.

Introduce a tetrad $\{(e_\mu)^b\}$, satisfying

$$(e_\mu)^c(e_\nu)_c = \eta_{\mu\nu}, \tag{5.34}$$

$$\eta^{\mu\nu}(e_\mu)_a(e_\nu)^b = \delta_a^b, \tag{5.35}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \tag{5.36}$$

where δ_a^b denotes the Kronecker symbol. It will be noted that the tetrad elements have combined 16 components, whereas the metric g_{ab} has 10 components. The relation

$$g_{ab} = (e_\mu)_a(e^\mu)_b \tag{5.37}$$

is non-unique by six degrees of freedom. This internal gauge degree of freedom is associated with the improper Poincaré group SO(3,1), describing rotations and boosts of the tetrad elements. In writing equations in tetrad form, we are led to insist on such Poincaré gauge invariance, in additional to general coordinate invariance.

Tetrad elements bring along connections, in terms of the one-forms

$$\omega_{a\mu\nu} = (e_\mu)^c \nabla_a (e_\nu)_c. \tag{5.38}$$

These connections may be used to define the gauge covariant derivative

$$\hat{\nabla}_a = \nabla_a + [\omega_a, \cdot], \tag{5.39}$$

whereby in particular

$$\hat{\nabla}_a (e_\mu)^b = 0. \tag{5.40}$$

Here, the commutator is defined by its action on tensors $\phi_{a_1 \cdots a_k \alpha_1 \cdots \alpha_l}$ as

$$[\omega_a, \phi_{a_1 \cdots a_k}]_{\alpha_1 \cdots \alpha_l} = \Sigma_i \omega_{a\alpha_i}{}^{\alpha_j} \omega_{a_1 \cdots a_k \alpha_1 \cdots \alpha_j \cdots \alpha_l}, \tag{5.41}$$

so that $[\omega_a, \omega_b]_{\mu\nu} = \omega_{a\mu}{}^\alpha \omega_{b\alpha\nu} - \omega_{a\nu}{}^\alpha \omega_{b\alpha\mu}$. The equations (5.30) and (5.31) become

$$\hat{\nabla}^a R_{ab\mu\nu} = 16\pi \tau_{b\mu\nu}, \tag{5.42}$$

wherein

$$R_{ab\mu\nu} = \nabla_a \omega_{b\mu\nu} - \nabla_b \omega_{a\mu\nu} + [\omega_a, \omega_b]_{\mu\nu}. \tag{5.43}$$

The tetrad elements satisfy the equations of structure

$$\partial_{[a} (e_\mu)_{b]} = (e^\nu)_{[b} \omega_{a]\nu\mu}. \tag{5.44}$$

It will be noted that the $\partial_t (e_\mu)_t$ are left un-defined in (5.46). Defining $\xi^b = (\partial_t)^b$, the four time-components introduce the tetrad lapse functions

$$N_\mu = (e_\mu)_a \xi^a \tag{5.45}$$

as freely specifiable functions. Thus, (5.46) becomes a system of ordinary differential equations

$$\partial_t (e_\mu)_b + \omega_{t\mu}{}^\nu (e_\nu)_b = \partial_b N_\mu + \omega_{b\mu}{}^\nu N_\nu. \tag{5.46}$$

The tetrad lapse functions are algebraically related to the familiar lapse N and shift functions N_p in the Hamiltonian formulation through

$$g_{at} = N_\alpha (e^\alpha)_a = (N_q N^q - N^2, N_p). \tag{5.47}$$

Summarizing, we have two representations of the Riemann tensor, in terms of the Christoffel connections and in terms of the Riemann-Cartan connections. The first gives rise to a representation in terms of second derivatives of the metric and, through the Einstein equations, gives rise to a second-order equation of motion. The second introduces a second-order equation of motion for the connections through (5.42), supplemented with the equations of structure (5.46) describing the evolution of the causal structure in the tangent bundle at each point.

5.5 The Weyl tensor

The Riemann tensor can be decomposed as the sum of a trace-free Weyl tensor C_{abcd} and remaining terms, involving the Ricci tensor and the scalar curvature tensor,

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} + g_{c[a}R_{b]d} - \frac{1}{3} g_{a[c}g_{d]b}R. \tag{5.48}$$

This applies to four-dimensional spacetime. In three-dimensions, we have $C_{abcd} \equiv 0$; in two dimensions we are left with the last term on the right hand-side of (5.48).

Exercises

1. Show that the integral $\int_\gamma u^a x^e ds$ is on the order of the surface area of the region enclosed by the curve γ.
2. Verify that the determinant of the metric satisfies $\sqrt{-g} = N\sqrt{h}$, by considering

$$\begin{pmatrix} N^c N_c - N^2 & N_j \\ N_i & h_{ij} \end{pmatrix} = \begin{pmatrix} 1 & N^i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -N^2 & 0 \\ 0 & h_{ij} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N^j & 1 \end{pmatrix}. \tag{5.49}$$

3. Verify the identity $(e_\mu)^c (e_\nu)^d \nabla^a R_{abcd} = \hat{\nabla}^a R_{ab\mu\nu}$.
4. Verify that the equations of structure obtain the system of ordinary

differential equations

$$\partial_t (e_\mu)_b + \omega_{t\mu\gamma}(e^\gamma)_b = \omega_{b\mu\gamma} N^\gamma + \partial_b N_\mu. \tag{5.50}$$

Interpret the connection $\omega_{t\mu\nu}$.