Chapter 8

A perfect fluid

The simplest fluid is dust: a fluid at zero temperature, and hence with zero viscosity and thermal conductivity. The stress-energy of dust is given by

$$T^{ab} = ru^b u^a, \tag{8.1}$$

where r denotes the density of the fluid as seen in the frame comoving with the fluid with velocity four-vector u^b. For example, one-dimensional motion of a perfect fluid along the $x-$axis introduces an energy density, and convection of energy and momentum

$$\rho = T^{tt}, \quad \dot{E} = T^{tx}, \quad \dot{P} = T^{xx}. \tag{8.2}$$

At finite temperature, pressure becomes nonzero and we consider

$$T^{ab} = rfu^a u^b + Pg^{ab}, \tag{8.3}$$

where we use the specific entropy

$$f = 1 + \frac{\gamma}{\gamma - 1}\frac{P}{r} \tag{8.4}$$

for a polytropic equation of state with polytropic index γ,

$$P = Kr^\gamma, \tag{8.5}$$

for some constant K. The specific enthalpy takes into account the mass-energy of both internal energy e and thermal pressure P, satisfying

$$P = (\gamma - 1)e. \tag{8.6}$$

The single fluid description (8.3) obtains by taking appropriate moments of the underlying momentum distribution of the particles. For particles of mass m, we have

$$r = \int d\mu_p, \;\; u^b = m^{-1}\int p^b d\mu_p, \;\;\; T^{ab} = m^{-1}\int p^a p^b d\mu_p. \tag{8.7}$$

where $d\mu_p = f(p^b)dp^x dp^y dp^z/p^t$ denotes the invariant measure for integration over momentum space. In this covariant description, the polytropic index is formally defined by through the definition of f in $rf = T^{ab}u_a u_b$.

In general, we have the first law of thermodynamics

$$dP = rdf - rTdS \tag{8.8}$$

in the presence of creation of entropy dS (per baryon) at a temperature T. The adiabatic law (8.5) is a special case with $dS = 0$ when K is constant. In the presence of shocks, entropy is created and K will vary along streamlines of the fluid.

A stress-energy tensor is subject to conservation of energy and momentum,

$$\partial_a T^{ab} = 0. \tag{8.9}$$

In the case of a perfect fluid, we further have conservation of baryon number

$$\partial_a(ru^a) = 0. \tag{8.10}$$

Together with the constraint $u^2 = -1$, (8.9) and (8.10) describe a partial differential-algebraic system of 6 equations in the six variables (u^b, r, P). There are 5 physical degrees of freedom; in the adiabatic limit, wherein K in (8.5) is constant, this reduces to four dynamical degrees of freedom.

We remark that in the computation of flows with shocks, entropy is created which changes K along streamlines. In the applications of some shock capturing schemes, we work with the full system of equations of six equations upon writing $\partial_a(\xi^a u^2) = 0$ to incorporate $u^2 = -1$ with $\xi^b = (1, 0, 0, 0)$ in the lab frame. Leaving the system in covariant form (with no reductions) permits covariant generalizations to ideal magnetohydrodynamics, for example.

At finite temperature, there obtains a finite sound speed. We can calculate the wave-structure of a one-dimensional perfect fluid somewhat analogously to the calculations on compressible gas dynamics in the Newtoniann limit. The energy equation $u_b\partial_a T^{ab}=0$ is automatically satisfied in the adiabatic limit (8.5). Consider, therefore, the momentum equation $v_b\partial_a T^{ab}=0$, where $v^b=(\sinh\lambda,\cosh\lambda)$ is orthogonal to u^b: $v_bu^b=0$. Together with $dP=rdf$, the momentum equation reduces to

$$\partial_a(fu^a)=0. \tag{8.11}$$

With baryon conservation (8.10), this obtains a system of two equations

$$\partial_a v^a + a_s v^a\partial_a\phi = 0, \tag{8.12}$$
$$\partial_a u^a + a_s^{-1} u^a\partial_a\phi = 0, \tag{8.13}$$

where we use $\phi=\int a_s r^{-1}dr$ and

$$a_s^2=\frac{rdf}{fdr}=\frac{dP}{fdr}. \tag{8.14}$$

Upon using $\partial_a v^a = u^a\partial_a\lambda$ and $\partial_a u^a = v^a\partial_a$, equations (8.15) can be combined by addition and subtraction, to arrive at the equations of motion in characteristic form

$$(u^a\pm a_s v^a)\nabla_a(\phi\pm\lambda)=0, \tag{8.15}$$

The structure (8.15) is that of two first-order, quasi-linear partial differential equations of the form

$$(\partial_t + w\partial_x)\psi = 0. \tag{8.16}$$

The quantity $\psi(t,x)=\Psi(x-wt)$ is a Riemann invariant along the directions $dx/dt=w$. In case of (8.15), the Riemann invariants are the combinations $R_\pm=\phi\pm\lambda$ along the characteristic directions

$$\left(\frac{dx}{dt}\right)_\pm = \frac{u^x\pm a_s v^x}{u^t\pm a_s v^t} = \frac{v\pm a_s}{1\pm va_s}. \tag{8.17}$$

In the comoving frame, where $u^b = (1,0,0,0)$ and $v^b = (0,1,0,0)$, the characteristic directions become

$$\left(\frac{dx}{dt}\right)_{\pm} = \pm a_s, \tag{8.18}$$

which shows that a_s denotes the adiabatic sound speed of the fluid.

It is of interest to also look at the non-relativistic limit, consisting of non-relativistic temperatures ($f \simeq 1$) and velocities ($\tanh\lambda \simeq \lambda$) to recover the familiar equations of compressible gas dynamics

$$(\partial_t \pm a_s\partial_x)\left(\frac{2a_s}{\gamma-1} + v\right) = 0. \tag{8.19}$$

The relativistic addition formula of parallel velocities (8.17) reduces to the Galilean transformation $(dx/dt)_{\pm} = v \pm a_s$.

Exercices

1. Derive (8.15) from the first law of thermodynamics $TdS = de + Pdr^{-1}$.
2. The characteristic form (8.15) is due to Taub (1948), originally using $\lambda = \ln\left(\frac{1+v}{1-v}\right)^{1/2}$. Verify this correspondence.
3. Use Schwarz's inequality on the definition $rf = T^{ab}u_au_b$ to show that $\gamma \leq 5/3$ (Taub, 1948). Note that $\gamma = 3/5$ is the Newtonian value of a monatomic gas.
4. Simple wave solutions are solutions in which one of the two Riemann invariants is constant throughout the fluid. Show that the special case of $\gamma = 3/2$ obtains $dx/dt = \tanh(5\lambda/4 - J/4)$ upon taking a constant Riemann invariant $R_+ = \lambda + \phi$. Plot the solution in response to initial data $\lambda(x) = \lambda_0 + \lambda_1\sin(2\pi x)$, using the method of characteristics, and describe the results.
5. Transverse magnetohydrodynamics describes a perfectly conducting fluid flowing along the $x-$direction with everywhere orthogonal magnetic field. It can be shown that the comoving specific magnetic field-strenght $\kappa = h/r$ is a conserved quantity, in view of $\partial_a(hu^a) = 0$. This can be incorporated through a modified equation of state, given by $P = Kr^\gamma + \kappa^2r^2$. Evaluate the magnetosonic sound speed.
6. The jump conditions of a gas about a shock front $\phi(t,x) = 0$ moving along the $x-$direction can be expressed covariantly in terms of $[F^b]\nu_b = 0$, where F^b is a covariant vector and $\nu_b = \partial_b\phi$ denotes the normal to the shock front. Apply this to T^{ab} and ru^b to derive the jump conditions. These are

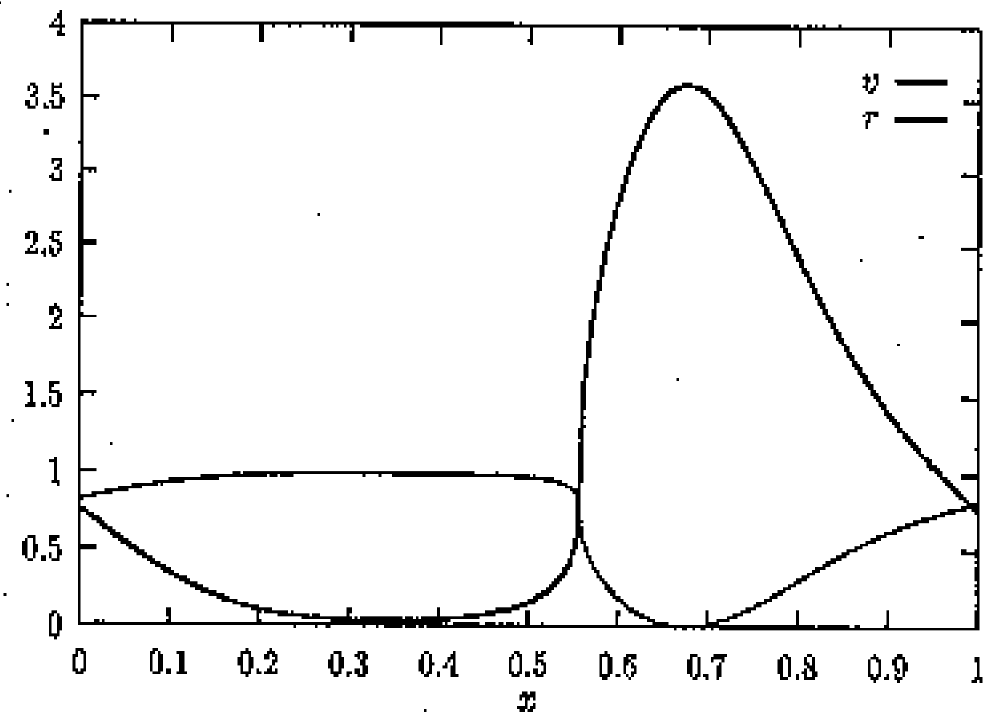

Figure 8.1: Shown is the velocity distribution v and rest mass density distribution r at the moment of breaking $t = t_B$. In this example $\lambda_0 = \lambda_1 = 7/5$, $J = 4.5$ and $t_B = 0.0963$. Reprinted from Maurice H.P.M. van Putten, Commun. Math. Phys., **141**, 63-77 (1991).

the relativistic Rankine-Hugoniot conditions.

7. Describe the relativistic Sod shock-tube problem in terms of simple waves, entropy and shock waves using the method of characteristics: the interaction of two fluids which are initially at rest in distinct thermodynamic states, e.g., the wave generated by suddenly removing a membrane between two fluids at different pressures and temperatures.

8. While Sod's shock-tube problem describes the instantaneous creation of a steady-state shock front, the shock formed by steepening of a simple wave as simulated in Fig. (8) emerges more gradually. Describe the evolution of the shock strength, and hence the rate of entropy creation, in the resulting shock as a function of time.