2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

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FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 10 - Solution

Instructor:	Prof. K. J. Bathe	Assigned:	05/06/2008
		Due:	05/13/2008

Problem 1 (10 points):

The governing differential equation is

$$\rho c_p \frac{d\theta}{dx} v = k \frac{d^2\theta}{dx^2}$$

The non-dimensional form is

$$\frac{d\Theta}{dX} = \frac{1}{Pe^e} \frac{d^2\Theta}{dX^2}$$

where

$$\Theta = \frac{\theta - \theta_L}{\theta_R - \theta_L}, \ X = \frac{x}{h} \text{ and } Pe^e = \frac{vh}{\alpha} = \frac{vh}{k/(\rho c_p)}$$

The principle of virtual temperature for a unit area is

$$\int \overline{\Theta} \frac{d\Theta}{dX} dX = \int \overline{\Theta} \frac{1}{Pe^e} \frac{d^2\Theta}{dX^2} dX = -\int \frac{1}{Pe^e} \frac{d\overline{\Theta}}{dX} \frac{d\Theta}{dX} dX + (boundary\ terms)$$

Therefore,

$$\int \left[\overline{\Theta} \frac{d\Theta}{dX} + \frac{1}{Pe^e} \frac{d\overline{\Theta}}{dX} \frac{d\Theta}{dX} \right] dX = (boundary\ terms)$$

If we use two-node elements, then for i-th element

$$\Theta^{(i)} = \frac{1}{2}(1-r)\Theta_i + \frac{1}{2}(1+r)\Theta_{i+1}$$

$$\frac{d\Theta^{(i)}}{dX} = \frac{d\Theta^{(i)}}{dr}\frac{dr}{dX} = \frac{d\Theta^{(i)}}{dr}\cdot\frac{2}{1} = -\Theta_i + \Theta_{i+1}$$

Hence

$$\begin{aligned}
&\int_0^1 \left[\bar{\Theta}^{(i)} \frac{d\Theta^{(i)}}{dX} + \frac{1}{Pe^e} \frac{d\bar{\Theta}^{(i)}}{dX} \frac{d\Theta^{(i)}}{dX} \right] dX \\
&= \int_{-1}^{+1} \left[\bar{\Theta}^{(i)} \frac{d\Theta^{(i)}}{dX} + \frac{1}{Pe^e} \frac{d\bar{\Theta}^{(i)}}{dX} \frac{d\Theta^{(i)}}{dX} \right] \frac{1}{2} dr \\
&= \hat{\bar{\Theta}}^{(i)^T} \left\{ \int_{-1}^{+1} \left(\begin{bmatrix} \frac{1}{2}(1-r) \\ \frac{1}{2}(1+r) \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \frac{1}{Pe^e} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \right) \frac{1}{2} dr \right\} \hat{\Theta}^{(i)} \\
&= \hat{\bar{\Theta}}^{(i)^T} \begin{bmatrix} -\frac{1}{2} + \frac{1}{Pe^e} & \frac{1}{2} - \frac{1}{Pe^e} \\ -\frac{1}{2} - \frac{1}{Pe^e} & \frac{1}{2} + \frac{1}{Pe^e} \end{bmatrix} \hat{\Theta}^{(i)}
\end{aligned}$$

where $\hat{\Theta}^{(i)} = \begin{bmatrix} \Theta_i & \Theta_{i+1} \end{bmatrix}^T$

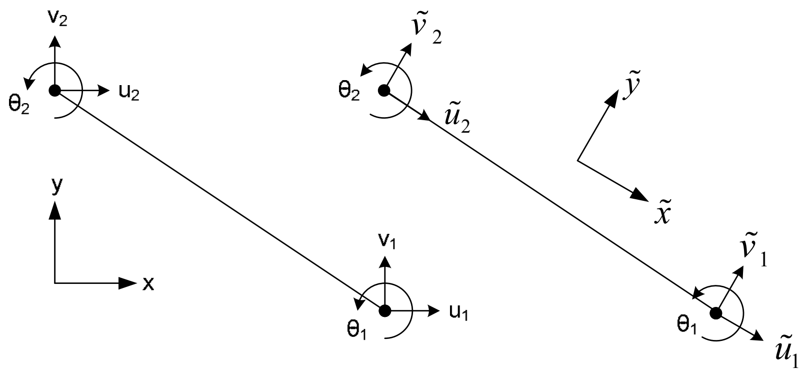
Therefore, the governing equation for the finite element node ***i*** by assembling the element ***(i-1)*** and ***(i)*** is

$$\begin{aligned}
&\left\{ \left(-\frac{1}{2} - \frac{1}{Pe^e} \right) \Theta_{i-1} + \left(\frac{1}{2} + \frac{1}{Pe^e} \right) \Theta_i \right\} + \left\{ \left(-\frac{1}{2} + \frac{1}{Pe^e} \right) \Theta_i + \left(\frac{1}{2} - \frac{1}{Pe^e} \right) \Theta_{i+1} \right\} \\
&= \left(-\frac{1}{2} - \frac{1}{Pe^e} \right) \Theta_{i-1} + \frac{2}{Pe^e} \Theta_i + \left(\frac{1}{2} - \frac{1}{Pe^e} \right) \Theta_{i+1} = 0
\end{aligned}$$

Finally,

$$\left(-1 - \frac{Pe^e}{2} \right) \Theta_{i-1} + 2\Theta_i + \left(\frac{Pe^e}{2} - 1 \right) \Theta_{i+1} = 0$$

Problem 2 (20 points):



$$\begin{bmatrix} \tilde{u}_k \\ \tilde{v}_k \end{bmatrix} = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}$$

We define strains and stresses in local coordinate system $(\tilde{x}, \tilde{y})$.

$$\tilde{u} = h_i \tilde{u}_i - \frac{st}{2} h_i \theta_i \text{ and } \tilde{v} = h_i \tilde{v}_i$$

where (r, s) is an iso-parametric coordinate system and t denotes the thickness.

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} = \frac{2}{L}\frac{\partial \tilde{u}}{\partial r} = \frac{2}{L}\frac{\partial h_i}{\partial r}\tilde{u}_i - \frac{st}{L}\frac{\partial h_i}{\partial r}\theta_i = \frac{2}{L}\frac{\partial h_i}{\partial r}\left(\frac{\sqrt{3}}{2}u_i - \frac{1}{2}v_i\right) - \frac{st}{L}\frac{\partial h_i}{\partial r}\theta_i$$

$$\frac{\partial \tilde{u}}{\partial \tilde{y}} = \frac{2}{t}\frac{\partial \tilde{u}}{\partial s} = -\frac{2}{t}\frac{t}{2}h_i\theta_i = -h_i\theta_i$$

$$\frac{\partial \tilde{v}}{\partial \tilde{x}} = \frac{2}{L}\frac{\partial \tilde{v}}{\partial r} = \frac{2}{L}\frac{\partial h_i}{\partial r}\tilde{v}_i = \frac{2}{L}\frac{\partial h_i}{\partial r}\left(\frac{1}{2}u_i + \frac{\sqrt{3}}{2}v_i\right)$$

$$\frac{\partial \tilde{v}}{\partial \tilde{y}} = \frac{2}{t}\frac{\partial \tilde{v}}{\partial s} = 0$$

Then,

$$\underline{\tilde{\varepsilon}} = \begin{bmatrix} \varepsilon_{\tilde{x}\tilde{x}} \\ \gamma_{\tilde{x}\tilde{y}} \\ \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \partial \tilde{u} / \partial \tilde{x} \\ \left(\partial \tilde{u} / \partial \tilde{y} + \partial \tilde{v} / \partial \tilde{x}\right)_{r=0} \\ u / x \end{bmatrix} = \underline{B}\underline{\hat{u}}$$

where $\hat{u} = \begin{bmatrix} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{bmatrix}^T$

Hence,

$$\underline{B} = \begin{bmatrix} \frac{\sqrt{3}}{2L} & -\frac{1}{2L} & -\frac{st}{2L} & -\frac{\sqrt{3}}{2L} & \frac{1}{2L} & \frac{st}{2L} \\ \frac{1}{2L} & \frac{\sqrt{3}}{2L} & -\frac{1}{2}(1+r)_{r=0} & -\frac{1}{2L} & -\frac{\sqrt{3}}{2L} & -\frac{1}{2}(1-r)_{r=0} \\ \frac{h_1}{x} & 0 & 0 & \frac{h_2}{x} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{20} & -\frac{1}{20} & -\frac{st}{20} & -\frac{\sqrt{3}}{20} & \frac{1}{20} & \frac{st}{20} \\ \frac{1}{20} & \frac{\sqrt{3}}{20} & -\frac{1}{2} & -\frac{1}{20} & -\frac{\sqrt{3}}{20} & -\frac{1}{2} \\ \frac{h_1}{x} & 0 & 0 & \frac{h_2}{x} & 0 & 0 \end{bmatrix}$$

where $x = h_1 x_1 + h_2 x_2 = \frac{1}{2}(1+r)(20) + \frac{1}{2}(1-r)(20 - 10\cos 30°) = \frac{40 - 5\sqrt{3}}{2} + \frac{5\sqrt{3}}{2} r$

The corresponding stress-strain matrix is (for a plane stress with a hoop strain)

$$\underline{C} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & 0 & \nu \\ 0 & \frac{1-\nu}{2} & 0 \\ \nu & 0 & 1 \end{bmatrix}$$