

MIT OpenCourseWare  
<http://ocw.mit.edu>

2.094 Finite Element Analysis of Solids and Fluids  
Spring 2008

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

**2.094**  
**FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS**  
**SPRING 2008**

**Homework 3 - Solution**

Instructor: Prof. K. J. Bathe

Assigned: 02/21/2008  
Due: 02/28/2008

---

**Problem 1 (10 points):**

From the plane strain condition,

$$\varepsilon_{zz} = \frac{1}{E} \{ \tau_{zz} - \nu(\tau_{xx} + \tau_{yy}) \} = 0$$
$$\therefore \tau_{zz} = \nu(\tau_{xx} + \tau_{yy})$$

The strains are

$$\varepsilon_{xx} = \frac{1}{E} \{ \tau_{xx} - \nu(\tau_{yy} + \tau_{zz}) \} = \frac{1}{E} \{ (1-\nu^2)\tau_{xx} - \nu(1+\nu)\tau_{yy} \} = 4.7667 \times 10^{-7}$$

$$\varepsilon_{yy} = \frac{1}{E} \{ \tau_{yy} - \nu(\tau_{xx} + \tau_{zz}) \} = \frac{1}{E} \{ (1-\nu^2)\tau_{yy} - \nu(1+\nu)\tau_{xx} \} = 4.3333 \times 10^{-8}$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} = 8.6667 \times 10^{-7}$$

The displacements in the four-node element can be written as

$$u(x, y) = a_0 + a_1x + a_2y + a_3xy$$

$$v(x, y) = b_0 + b_1x + b_2y + b_3xy$$

(\*Note that you can also use the following form of displacements as in the class.

$$u(x, y) = h_1(x, y)u_1 + h_2(x, y)u_2 + h_3(x, y)u_3 + h_4(x, y)u_4$$

$$v(x, y) = h_1(x, y)v_1 + h_2(x, y)v_2 + h_3(x, y)v_3 + h_4(x, y)v_4$$

But, then equations are slightly complicate to solve. Please see attached sample solution which uses this method)

Then,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_1 + a_3 y = 4.7667 \times 10^{-7}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = b_2 + b_3 x = 4.3333 \times 10^{-8}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + a_3 x + b_1 + b_3 y = 8.6667 \times 10^{-7}$$

Since these equations hold for all x and y,

$$a_3 = b_3 = 0$$

$$a_1 = 4.7667 \times 10^{-7}$$

$$b_2 = 4.3333 \times 10^{-8}$$

$$a_2 + b_1 = 8.6667 \times 10^{-7}$$

Therefore,

$$u(x, y) = a_0 + 4.7667 \times 10^{-7} x + a_2 y$$

$$v(x, y) = b_0 + b_1 x + 4.3333 \times 10^{-8} y$$

The boundary conditions with setting the node 3 to the origin of the coordinate system are

$$u(0, 0) = a_0 = 0$$

$$v(0, 0) = b_0 = 0$$

$$v(5, 0) = b_0 + 5b_1 = 0$$

Finally,

$$u(x, y) = 4.7667 \times 10^{-7} x + 8.6667 \times 10^{-7} y$$

$$v(x, y) = 4.3333 \times 10^{-8} y$$

The displacements at each node are

	u (in.)	v (in.)
Node 1 (x=5, y=8)	$9.3167 \times 10^{-6}$	$3.4666 \times 10^{-7}$
Node 2 (x=0, y=8)	$6.9334 \times 10^{-6}$	$3.4666 \times 10^{-7}$
Node 3 (x=0, y=0)	0	0
Node 4 (x=5, y=0)	$2.3833 \times 10^{-6}$	0

**Problem 2 (20 points):**

(a) The stresses can be obtained from a bilinear interpolation.

$$\begin{aligned}\tau_{xx}^{(4)} &= \frac{1}{4}(1+x)(1+y)(298.5) + \frac{1}{4}(1-x)(1+y)(624.0) + \frac{1}{4}(1-x)(1-y)(1.146) + \frac{1}{4}(1+x)(1-y)(-324.3) \\ &= 149.8365 - 162.7365x + 311.4135y - 0.0135xy\end{aligned}$$

$$\begin{aligned}\tau_{yy}^{(4)} &= \frac{1}{4}(1+x)(1+y)(-914.9) + \frac{1}{4}(1-x)(1+y)(169.9) + \frac{1}{4}(1-x)(1-y)(-16.96) + \frac{1}{4}(1+x)(1-y)(-1102) \\ &= -465.9900 - 542.4600x + 93.4900y + 0.0600xy\end{aligned}$$

$$\begin{aligned}\tau_{xy}^{(4)} &= \frac{1}{4}(1+x)(1+y)(-370.8) + \frac{1}{4}(1-x)(1+y)(-588.8) + \frac{1}{4}(1-x)(1-y)(-209.1) + \frac{1}{4}(1+x)(1-y)(8.865) \\ &= -289.9587 + 108.9912x - 189.8412y + 0.0087xy\end{aligned}$$

Note that the coefficients of 'xy' in each stress must be zero because of our displacement assumption. But here we have them due to rounding. (The strains do not have 'xy' terms because they are derivatives of the displacements.)

$$B^{(4)} = \frac{1}{4} \begin{bmatrix} (1+y) & -(1+y) & -(1-y) & (1-y) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1+x) & (1-x) & -(1-x) & -(1+x) \\ (1+x) & (1-x) & -(1-x) & -(1+x) & (1+y) & -(1+y) & -(1-y) & (1-y) \end{bmatrix}$$

$$F^{(4)} = \int_{-1}^1 \int_{-1}^1 B^{(4)T} \begin{bmatrix} \tau_{xx}^{(4)} \\ \tau_{yy}^{(4)} \\ \tau_{xy}^{(4)} \end{bmatrix} (0.1) dx dy$$

$$= \begin{bmatrix} 0.00127 \\ -57.99301 \\ 28.02571 \\ 29.96603 \\ -100.0049 \\ 6.806910 \\ 51.18483 \\ 42.01317 \end{bmatrix}$$

(b) Consider element 4.

a. Horizontal equilibrium:

$$0 - 57.99 + 28.03 + 29.97 \cong 0$$

b. Vertical equilibrium:

$$-100 + 6.81 + 51.18 + 42.01 = 0$$

c. Moment equilibrium about its local node 3:

$$-100 \times 2 + 57.99 \times 2 + 42.01 \times 2 = 0$$

**Problem 3 (10 points):**

For the element A,

$$\{u_1, v_1, u_4, v_4\}^T = \{U_1, U_2, U_3, U_4\}^T$$

The corresponding components of the stiffness matrix are

$$\hat{K}_A = \begin{bmatrix} a_{11} & a_{12} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{27} & a_{28} \\ a_{71} & a_{72} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{87} & a_{88} \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{matrix}$$

For the element B,

$$\{u_1, v_1, \theta_1\}^T = \{U_3, U_4, U_5\}^T$$

The corresponding components of the stiffness matrix are

$$\hat{K}_B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} U_3 \\ U_4 \\ U_5 \end{matrix}$$

Then the global stiffness matrix is

$$K = \begin{bmatrix} a_{11} & a_{12} & a_{17} & a_{18} & 0 \\ a_{21} & a_{22} & a_{27} & a_{28} & 0 \\ a_{71} & a_{72} & a_{77} + b_{11} & a_{78} + b_{12} & b_{13} \\ a_{81} & a_{82} & a_{87} + b_{21} & a_{88} + b_{22} & b_{23} \\ 0 & 0 & b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{matrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{matrix}$$