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2.094 Finite Element Analysis of Solids and Fluids  
Spring 2008

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# 2.094

## FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

### SPRING 2008

### Homework 4 - Solution

Instructor: Prof. K. J. Bathe	Assigned: 02/28/2008	Due: 03/06/2008
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**Problem 1 (20 points):**

(a)

$$h_5(x, y) = \frac{1}{24}(3-x)(4-y^2);$$

$$h_6(x, y) = \frac{1}{24}(3+x)(4-y^2)$$

$$h_1(x, y) = \frac{1}{24}(3+x)(2+y) - \frac{1}{2}h_6(x, y);$$

$$h_2(x, y) = \frac{1}{24}(3-x)(2+y) - \frac{1}{2}h_5(x, y)$$

$$h_3(x, y) = \frac{1}{24}(3-x)(2-y) - \frac{1}{2}h_5(x, y);$$

$$h_4(x, y) = \frac{1}{24}(3+x)(2-y) - \frac{1}{2}h_6(x, y)$$

(b) Note that above interpolation functions satisfy  $\sum_{i=1}^6 h_i(x, y) = 1$

\* Rigid body translation in x-direction

$$u_1 = u_2 = \dots = u_6 = 2.0 \rightarrow u(x, y) = \sum_{i=1}^6 h_i(x, y)u_i = 2.0 \sum_{i=1}^6 h_i(x, y) = 2.0$$

$$\rightarrow \underline{\varepsilon}^T = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}] = \left[ \frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] = \underline{0}$$

\* Rigid body translation in y-direction

$$v_1 = v_2 = \dots = v_6 = 2.0 \rightarrow v(x, y) = \sum_{i=1}^6 h_i(x, y)v_i = 2.0 \sum_{i=1}^6 h_i(x, y) = 2.0$$

$$\rightarrow \underline{\varepsilon}^T = [\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}] = \left[ \frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] = \underline{0}$$

\* Rigid body rotation by  $60^\circ$

Here we use:

$$u_i = -\theta \cdot y_i \text{ and } v_i = \theta \cdot x_i \quad (\theta = 60^\circ)$$

Hence

$$u(x, y) = \sum_{i=1}^6 h_i(x, y) \{-\theta \cdot y_i\} = -\theta \cdot y, \quad v(x, y) = \sum_{i=1}^6 h_i(x, y) \{\theta \cdot x_i\} = \theta \cdot x$$

which corresponds to the rigid body rotation. Also

$$\underline{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \underline{0}$$

Note: The strains used in the class up to now correspond to infinitesimally small displacements and strains. Therefore we need to use the kinematics corresponding to these assumptions meaning  $\cos \theta = 1$  and  $\sin \theta = \theta$ .

Later on we will allow for large deformations where we need to use  $\cos \theta$  and  $\sin \theta$ , but then we also need to use the large strain theory.

(c)

$$\underline{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \end{bmatrix}$$

$$\underline{H}^S = \underline{H}|_{x=-3} = \begin{bmatrix} 0 & \frac{1}{8}y(y+2) & \frac{1}{8}y(y-2) & 0 & \frac{1}{4}(4-y^2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}y(y+2) & \frac{1}{8}y(y-2) & 0 & \frac{1}{4}(4-y^2) & 0 \end{bmatrix}$$

$$\underline{f}^S = \begin{bmatrix} P \\ 0 \end{bmatrix}$$

$$\underline{R}_S = \int_S (\underline{H}^S)^T \underline{f}^S dS = \int_{-2}^2 (\underline{H}^S)^T \underline{f}^S (0.1) dy$$

$$\underline{R}_S^T = (0.1P) \begin{bmatrix} 0 & \frac{2}{3} & \frac{2}{3} & 0 & \frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Problem 2 (20 points):**

(a)

$$x = \sum_{i=1}^4 h_i(r, s)x_i = \frac{1}{4}(3 + 13r + s + 3rs)$$

$$y = \sum_{i=1}^4 h_i(r, s)y_i = (1 + 3s + rs)$$

Therefore,

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 13 + 3s & 4s \\ 1 + 3r & 12 + 4r \end{bmatrix}$$

(b)

$$\underline{J}^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} = \frac{1}{39 + 13r + 8s} \begin{bmatrix} 12 + 4r & -4s \\ -1 - 3r & 13 + 3s \end{bmatrix}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \varepsilon_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{u}{(x+7)} \end{bmatrix} = \underline{B}\hat{u}$$

Therefore,

$$\underline{B}_{u_1} = \begin{bmatrix} \frac{\partial h_1}{\partial x} \\ 0 \\ \frac{\partial h_1}{\partial y} \\ \frac{h_1}{x+7} \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial h_1}{\partial x} &= \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial h_1}{\partial s} \frac{\partial s}{\partial x} = \frac{1+s}{4} \cdot \frac{12+4r}{39+13r+8s} - \frac{1+r}{4} \cdot \frac{4s}{39+13r+8s} = \frac{3+r+2s}{39+13r+8s} \\ \frac{\partial h_1}{\partial y} &= \frac{\partial h_1}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial h_1}{\partial s} \frac{\partial s}{\partial y} = -\frac{1+s}{4} \cdot \frac{1+3r}{39+13r+8s} + \frac{1+r}{4} \cdot \frac{13+3s}{39+13r+8s} = \frac{6+5r+s}{2(39+13r+8s)} \\ \frac{h_1}{x+7} &= \frac{(1+r)(1+s)}{31+13r+s+3rs} \end{aligned}$$