2.094 Finite Element Analysis of Solids and Fluids
Spring 2008

2.094

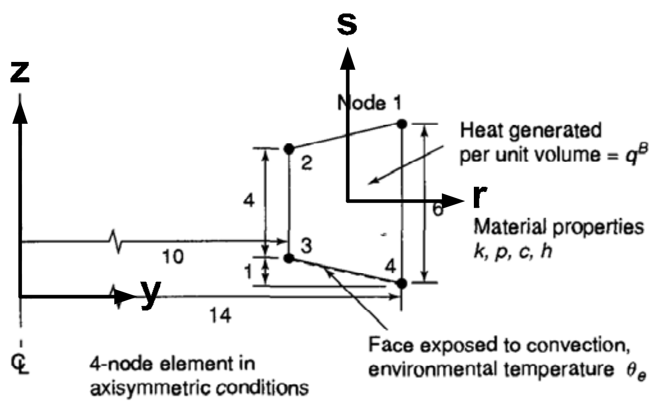
FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 9 - Solution

		Assigned:	04/17/2008
Instructor:	Prof. K. J. Bathe	Due:	04/24/2008

Problem 1 (10 points):



$$\underline{H} = \frac{1}{4}\left[(1+r)(1+s) \quad (1-r)(1+s) \quad (1-r)(1-s) \quad (1+r)(1-s)\right]$$

$$\underline{H}^S = \underline{H}\big|_{s=-1} = \begin{bmatrix} 0 & 0 & \frac{(1-r)}{2} & \frac{(1+r)}{2} \end{bmatrix}$$

$$\underline{J} = \begin{bmatrix} 2 & \frac{s}{2} \\ 0 & \frac{5+r}{2} \end{bmatrix} \text{ from } y = 2(6+r) \text{ and } z = \frac{1}{2}(6+5s+rs)$$

$$\det \underline{J} = 5+r$$

$$\underline{B} = \frac{1}{5+r}\begin{bmatrix} \frac{5+r}{2} & -\frac{s}{2} \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1+s}{4} & -\frac{1+s}{4} & -\frac{1-s}{4} & \frac{1-s}{4} \\ \frac{1+r}{4} & \frac{1-r}{4} & -\frac{1-r}{4} & -\frac{1+r}{4} \end{bmatrix}$$

$$\underline{k} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

The conductivity matrix is

$$\underline{K}^k = \int_V \underline{B}^T \underline{k}\, \underline{B}\, y dy dz = \int_{-1}^{+1}\int_{-1}^{+1} \underline{B}^T \underline{k}\, \underline{B} (5+r)\left\{2(6+r)\right\} dr ds$$

The convection matrix is

$$\underline{K}^c = \int_{S_c} h \underline{H}^{sT} \underline{H}^s dS = \int_{-1}^{+1} h \underline{H}^{sT} \underline{H}^s \left(\frac{\sqrt{17}}{2}\right)\left\{2(6+r)\right\} dr$$

because $dS = y dl = y \det \underline{J}^s dr$

where $\det \underline{J}^s = \left[\left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2\right]^{1/2}\Bigg|_{s=-1} = \frac{\sqrt{17}}{2}$ with $\frac{\partial y}{\partial r} = 2$ and $\frac{\partial z}{\partial r} = \frac{s}{2}$

The heat flow load vector is

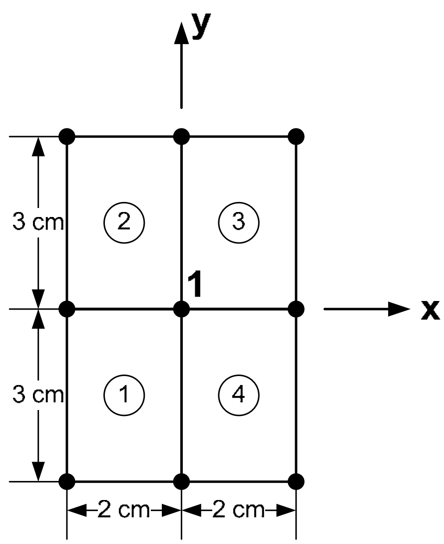
$$\underline{Q} = \underline{Q}_B + \underline{Q}^e$$

where

$$\underline{Q}_B = \int_V \underline{H}^T q^B dV = \int_{-1}^{+1}\int_{-1}^{+1} \underline{H}^T q^B \left\{2(6+r)\right\}(5+r) dr ds$$

$$\underline{Q}^e = \int_{S_e} h \underline{H}^{sT} \underline{H}^s \underline{\hat{\theta}}_e dS = \int_{-1}^{+1} h \underline{H}^{sT} \underline{H}^s \underline{\hat{\theta}}_e \left\{2(6+r)\right\}\left(\frac{\sqrt{17}}{2}\right) dr$$

Problem 2 (10 points):



Note that ϕ is zero on the boundary of the shaft. Therefore we have only one degree of freedom at node 1, ϕ_1.

Because of the symmetry, we need to consider only one element.

We have for element 1,

$$\phi = \frac{1}{4}(1+r)(1+s)\phi_1 = \underline{H}^{(1)}\underline{\hat{\phi}}$$

$$\phi' = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{2}{3}\frac{\partial \phi}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(1+s) \\ \frac{1}{6}(1+r) \end{bmatrix} \left[\phi_1\right] = \underline{B}^{(1)}\underline{\hat{\phi}}$$

Then FE governing equation for a unit length of the shaft is

$$\left\{ 4\int_{-1}^{+1}\int_{-1}^{+1} \begin{bmatrix} \frac{1}{4}(1+s) & \frac{1}{6}(1+r) \end{bmatrix} \begin{bmatrix} \frac{1}{2G} & 0 \\ 0 & \frac{1}{2G} \end{bmatrix} \begin{bmatrix} \frac{1}{4}(1+s) \\ \frac{1}{6}(1+r) \end{bmatrix} \det J dr ds \right\} \phi_1$$

$$= 4\theta \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{4}(1+r)(1+s)\det J dr ds$$

where $\det J = \frac{3}{2}$

Hence,

$$\phi_1 = \frac{54}{13} G\theta$$

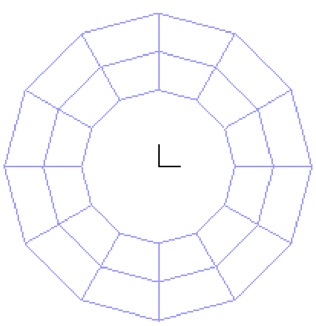
Thus we obtain

$$T = 4\int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{2}(1+r)(1+s)\left(\frac{54}{13}G\theta\right)\det J dr ds = \frac{648}{13}G\theta$$

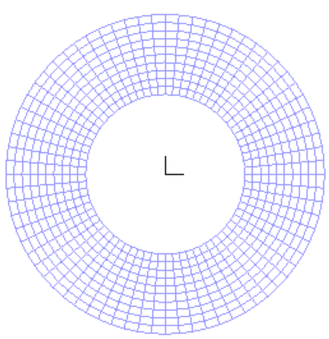
so that

$$\frac{T}{\theta} = \frac{648}{13}G$$

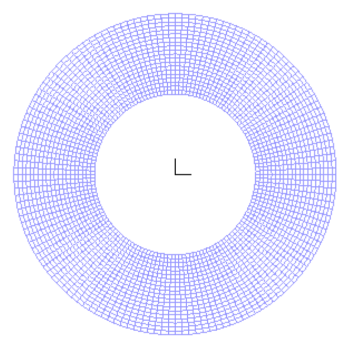
Problem 3 (10 points):



(1) coarse mesh (2x12)



(2) fine mesh (10x72)



(3) finest mesh (20x144)

The analytical velocity and pressure distributions are,

$$v_\theta(r) = Ar + \frac{B}{r} \text{ where } A = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2} \text{ and } B = \frac{\left(\omega_1 - \omega_2\right) r_1^2 r_2^2}{r_2^2 - r_1^2}$$

$$p(r) = \rho\left(\frac{1}{2}A^2r^2 - \frac{1}{2}\frac{B^2}{r^2} + 2AB\ln(r)\right) + C$$

where

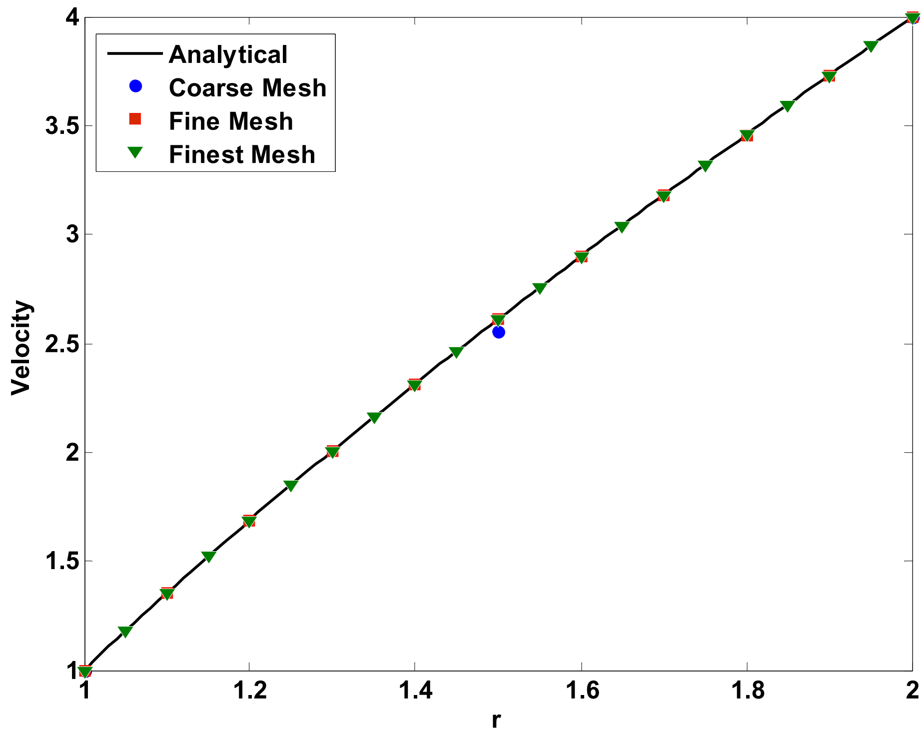
$$C = p\big|_{r=r_1} - \rho\left(\frac{1}{2}A^2r_1^{\,2} - \frac{1}{2}\frac{B^2}{r_1^{\,2}} + 2AB\ln(r_1)\right)$$

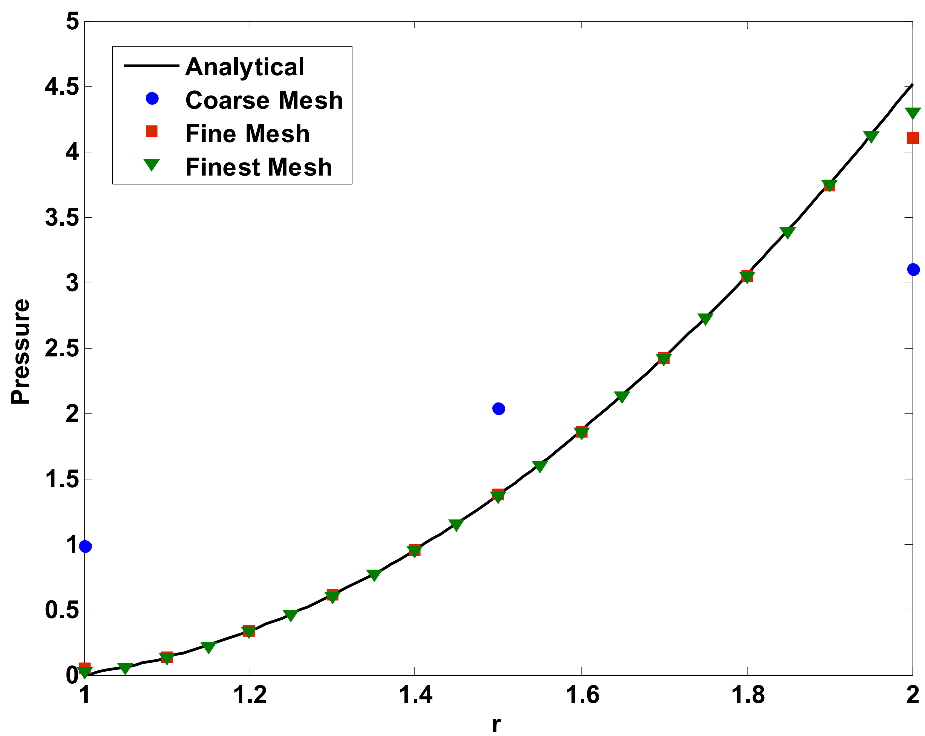
In this problem,

$$A = \frac{(2)(2^2) - (1)(1^2)}{2^2 - 1^2} = \frac{7}{3},\ B = \frac{(1-2)(1^2)(2^2)}{2^2 - 1^2} = -\frac{4}{3}$$

$$C = 0 - (1)\left(\frac{1}{2}(7/3)^2(1^2) - \frac{1}{2}\frac{(-4/3)^2}{1^2} + 2\frac{7}{3}\left(-\frac{4}{3}\right)\ln(1)\right) = -\frac{11}{6}$$

The results are compared with these analytical solutions in the following figures. Note that we obtained the velocities very close to the analytical ones even with the coarse mesh. However the pressure deviates from the analytical solution on the boundaries because the pressure is constant in the element we used and we averaged the pressure at each node for the plot.


Analytical
Coarse Mesh
Fine Mesh
Finest Mesh
Velocity
r


Analytical
Coarse Mesh
Fine Mesh
Finest Mesh
Pressure
r