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2.094 Finite Element Analysis of Solids and Fluids  
Spring 2008

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# 2.094

## FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

### SPRING 2008

### Quiz #2 - Solution

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#### Problem 1 (10 points)

$$a) \quad {}^t_0\underline{X} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 7/6 & 0 \\ 0 & 8/9 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{3}}{12} & -\frac{4}{9} \\ \frac{7}{12} & \frac{4\sqrt{3}}{9} \end{bmatrix}$$

$$b) \quad {}^t_0\underline{\varepsilon} = \frac{1}{2} ({}^t_0\underline{X}^T {}^t_0\underline{X} - \underline{I}) = \frac{1}{2} ({}^t_0\underline{U}^2 - \underline{I}) = \begin{bmatrix} \frac{13}{72} & 0 \\ 0 & -\frac{17}{162} \end{bmatrix}$$

$${}^t_0\underline{S} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{pmatrix} \frac{13}{72} \\ -\frac{17}{162} \\ 0 \end{pmatrix} = E \begin{pmatrix} \frac{13}{72} \\ -\frac{17}{162} \\ 0 \end{pmatrix}$$

c)

$${}^t_0\underline{\tau} = \frac{{}^t_0\rho}{{}_0\rho} {}^t_0\underline{X} {}^t_0\underline{S} {}^t_0\underline{X}^T = \frac{1}{\det {}^t_0\underline{X}} {}^t_0\underline{X} {}^t_0\underline{S} {}^t_0\underline{X}^T = \frac{1}{\frac{7\sqrt{3}}{12} \cdot \frac{4\sqrt{3}}{9} + \frac{4}{9} \cdot \frac{7}{12}} \begin{bmatrix} \frac{7\sqrt{3}}{12} & -\frac{4}{9} \\ \frac{7}{12} & \frac{4\sqrt{3}}{9} \end{bmatrix} \begin{bmatrix} \frac{13}{72} E & 0 \\ 0 & -\frac{17}{162} E \end{bmatrix} \begin{bmatrix} \frac{7\sqrt{3}}{12} & \frac{7}{12} \\ -\frac{4}{9} & \frac{4\sqrt{3}}{9} \end{bmatrix}$$

**Problem 2 (10 points)**

The governing equation we need is

$$\int_V \bar{e}_{ij} \tau_{ij} dV = \mathfrak{R}$$

By substituting  $\tau_{ij} = -p\delta_{ij} + \mu(v_{i,j} + v_{j,i}) = -p\delta_{ij} + 2\mu e_{ij}$ ,

$$\int_V \bar{e}_{ij} \tau_{ij} dV = \int_V \bar{e}_{ij} (2\mu) e_{ij} dV - \int_V \bar{e}_{ii} p dV$$

In the following calculations, we need to know the interpolation function and its derivatives only at the node 1.

$$h_1 = \frac{1}{2} x_1 (1 + x_1) x_2 (1 + 2x_2)$$

$$h_{1,1} = \frac{1}{2} (1 + 2x_1) x_2 (1 + 2x_2)$$

$$h_{1,2} = \frac{1}{2} x_1 (1 + x_1) (1 + 4x_2)$$

The volume integration of the element assuming the unit depth into the  $x_3$  direction is,

$$\int_V ( ) dV = \int_{-1}^{+1} \int_{-1/2}^{+1/2} ( ) dx_1 dx_2$$

a) Evaluate  $K(1,1)$

Let

$$\underline{e} = \begin{pmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{pmatrix} = \begin{pmatrix} v_{1,1} \\ v_{2,2} \\ v_{1,2} + v_{2,1} \end{pmatrix} = \underline{B} \hat{v} \quad \text{where } \hat{v}^T = [v_1^1 \quad v_2^1 \quad v_1^2 \quad v_2^2 \quad \dots \quad v_1^9 \quad v_2^9]$$

Then,

$$\int_V \bar{e}_{ij} (2\mu) e_{ij} dV = \hat{v}^T \left( \int_V \underline{B}^T \underline{C} \underline{B} dV \right) \hat{v} \quad \text{where } \underline{C} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

We only need the first column of  $\underline{B}$  to compute  $K(1,1)$ ,

$$\underline{B}_{(1)}^T = [h_{1,1} \quad 0 \quad h_{1,2}]$$

Therefore,

$$K(1,1) = \int_V \underline{B}_{(1)}^T \underline{C} \underline{B}_{(1)} dV = \int_V \left\{ 2\mu (h_{1,1})^2 + \mu (h_{1,2})^2 \right\} dV$$

\*\*\* Or, simply from Equation 7.79 in the textbook,

$$K_{\mu v_1 v_1} = \int_V \left( 2\mu H_{,x_1}^T H_{,x_1} + \mu H_{,x_2}^T H_{,x_2} \right) dV$$

$$K(1,1) = K_{\mu v_1 v_1}(1,1) = \int_V \left\{ 2\mu (h_{1,1})^2 + \mu (h_{1,2})^2 \right\} dV$$

b) Evaluate  $K(1,21)$

Let

$$\underline{e}_{ii} = \underline{e}_{11} + \underline{e}_{22} = \underline{v}_{1,1} + \underline{v}_{2,2} = \underline{B}_v \underline{\hat{v}}$$

$$p = p_0 + p_1 x_1 + p_2 x_2 = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \underline{\tilde{H}} \underline{\hat{p}}$$

Then,

$$-\int_V \underline{e}_{ii} p dV = \underline{\hat{v}}^T \left( -\int_V \underline{B}_v^T \underline{\tilde{H}} dV \right) \underline{\hat{p}}$$

We need the first column of  $\underline{B}_v$  and the third column of  $\underline{\tilde{H}}$  to compute  $K(1,21)$ ,

$$\underline{B}_{v(1)}^T = [h_{1,1}] \text{ and } \underline{\tilde{H}}_{(3)} = [x_2]$$

Therefore,

$$K(1,21) = -\int_V \underline{B}_{v(1)}^T \underline{\tilde{H}}_{(3)} dV = -\int_V h_{1,1} x_2 dV$$

\*\*\* Or, simply from Equation 7.81 in the textbook,

$$K_{v_1 p} = -\int_V H_{,x_1}^T \underline{\tilde{H}} dV$$

$$K(1,21) = K_{v_1 p}(1,3) = -\int_V h_{1,1} x_2 dV$$

c) The 9/3 element is a suitable element for the incompressible analysis because it does not lock and passes the inf-sup condition and therefore it shows an optimal convergence.