Numerical Marine Hydrodynamics

- Finite Element and Spectral Methods
  - Galerkin Methods
  - Computational Galerkin Methods
    - Spectral Methods
    - Finite Element Method
  - Finite Element Methods
    - Ordinary Differential Equation
    - Partial Differential Equations
    - Complex geometries

Partial Differential Equations

Quasi-linear PDE

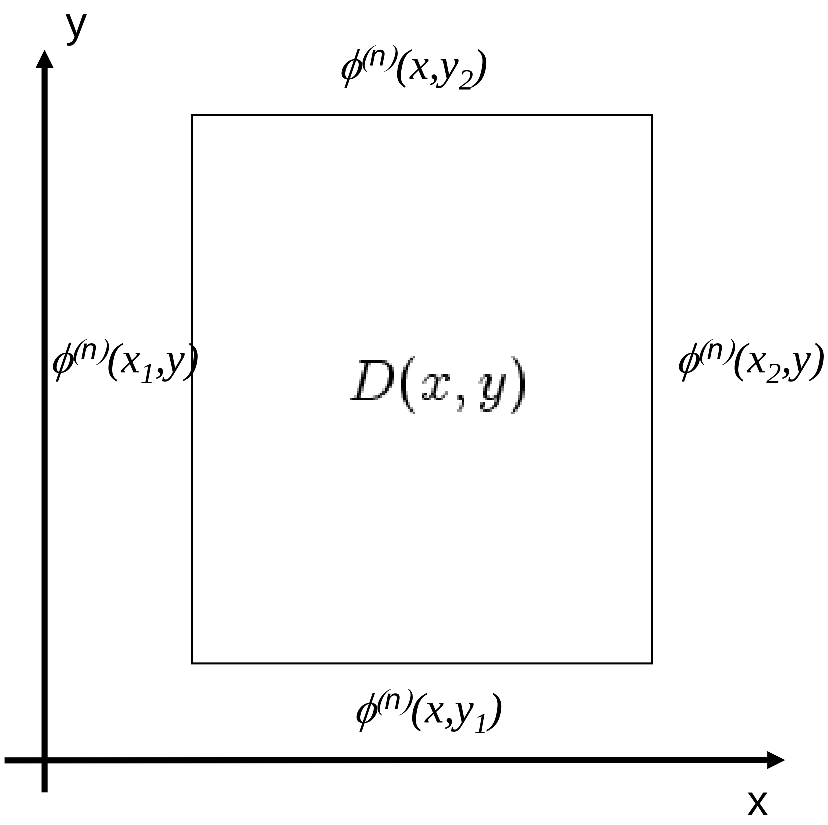
$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = F(x, y, \phi, \phi_x, \phi_y)$$

A,B and C Constants

$B^2 - 4AC > 0$ Hyperbolic

$B^2 - 4AC = 0$ Parabolic

$B^2 - 4AC < 0$ Elliptic



Galerkin's Method

Differential Equation

$$L(u) = 0$$

Boundary Conditions

$$S(u) = 0$$

Test Function Solution

$$\tilde{u} = u_0(x,y) + \sum_{j=1}^{N} a_j \phi(x,y)$$

Remainder

$$R(u_0, a_1, \ldots a_N, x, y) = L(\tilde{u}) = L(u_0) + \sum_{j=1}^{N} a_j L(\phi(x,y))$$

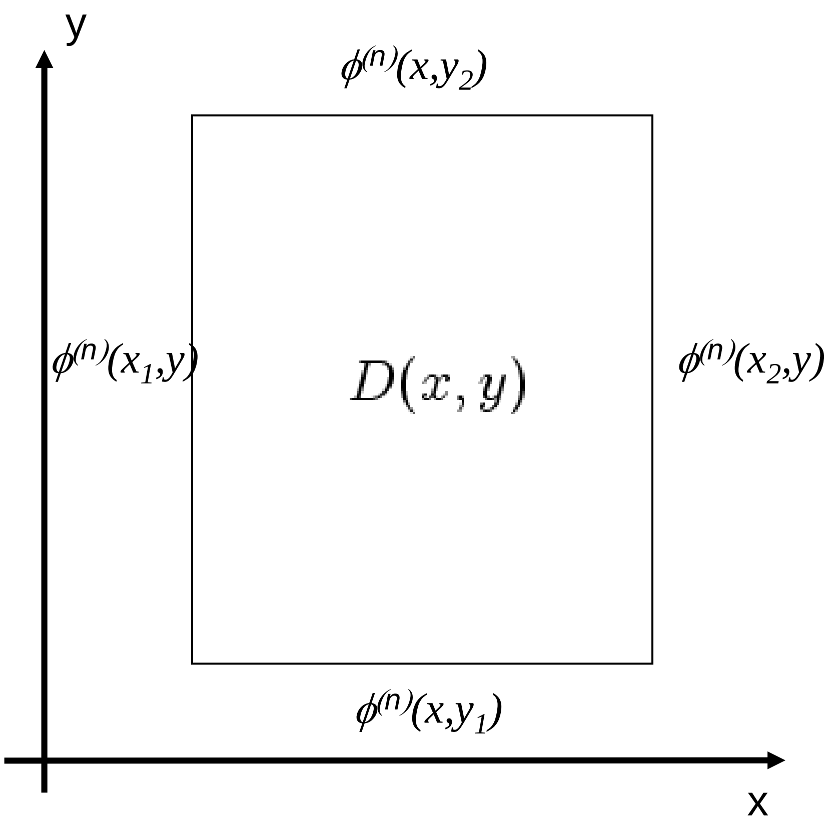
Inner Product

$$(f,g) = \int\int_D fg\,dx\,dy$$

Galerkin's Method

$$(R, \phi_k) = 0$$

$$\sum_{j=1}^{N} a_j (L(\phi_j), \phi_k) = -(L(u_0), \phi_k)$$



Galerkin's Method Example

Differential Equation

$$\frac{dy}{dx} - y = 0$$

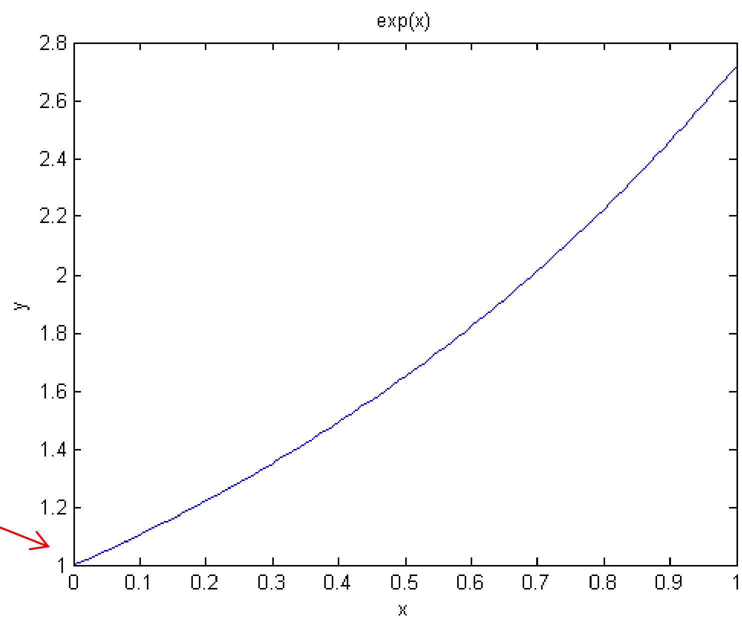
Boundary Conditions

$$y = 1\ , \quad x = 0$$

Power Series

$$\tilde{y} = 1 + \sum_{j=1}^{N} a_j x^j$$

Boundary Condition



Alternative

$$\tilde{y} = \sum_{j=0}^{N} a_j x^j$$

$$a_0 = 1$$

Galerkin's Method
Example

Remainder

$$R = -1 + \sum_{j=1}^{N} a_j (j x^{j-1} - x^j)$$

Complete Test Function Set

$$(R, x^{k-1}) = 0 \; , \quad k = 1, \ldots N$$

Algebraic Equations

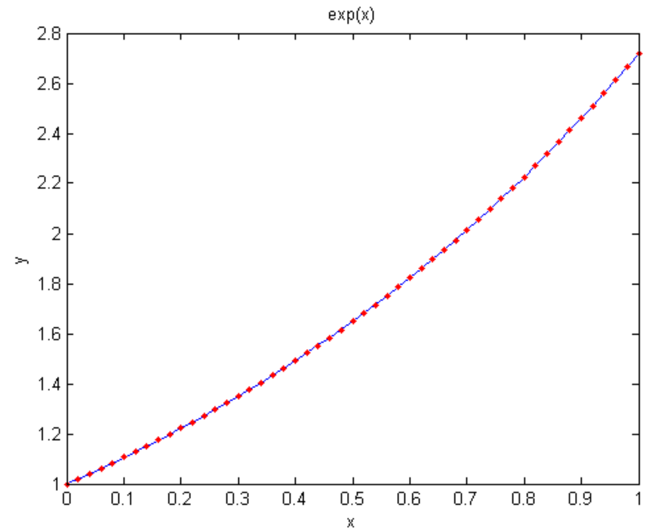
$$\mathbf{Ma} = \mathbf{d}$$

$$d_k = (1, x^{k-1})$$

$$m_{kj} = (j x^{j-1} - x_j, x^{k-1}) = \frac{j}{j+k-1} - \frac{1}{j+k}$$

exp_eq.m

```
N=3;
d=zeros(N,1);
m=zeros(N,N);
for k=1:N
    d(k)=1/k;
    for j=1:N
        m(k,j) = j/(j+k-1)-1/(j+k);
    end
end
a=inv(m)*d;
y=ones(1,n);
for k=1:N
    y=y+a(k)*x.^k
end
```



Galerkin's Method Example

$$N = 3$$

$$\mathbf{a}^T = [1.0141, 0.4225, 0.2817];$$

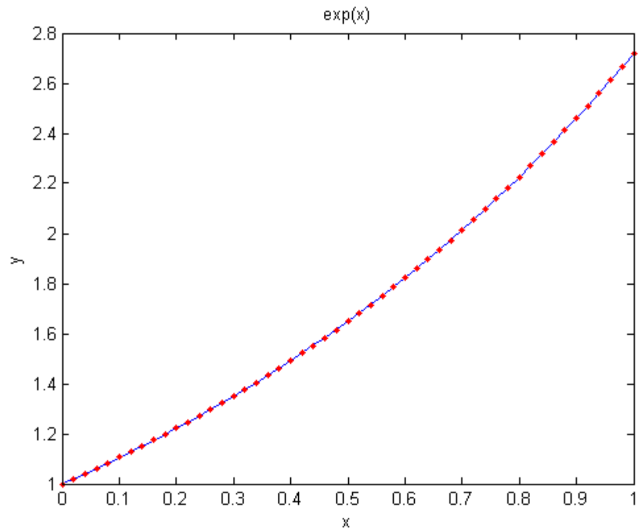
$$\tilde{y} = 1 + 1.0141x + 0.4225x^2 + 0.2817x^3$$

L_2 Error

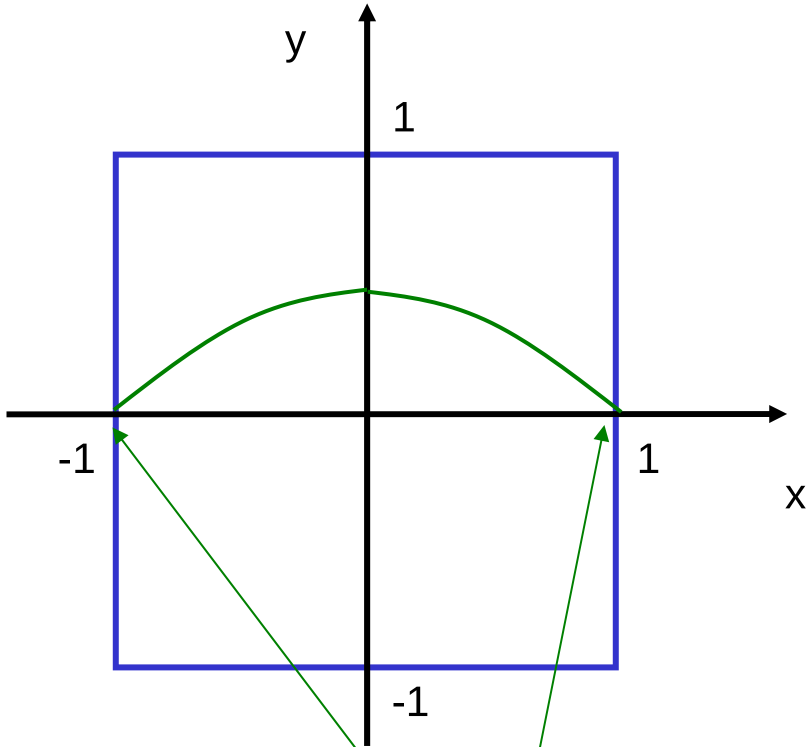
$$L_2 = ||y - \tilde{y}||_2 = \sqrt{\sum_{\ell=1}^{L} (y(x_\ell) - \tilde{y}(x_\ell))^2}$$

```
N=3;
d=zeros(N,1);
m=zeros(N,N);
for k=1:N
    d(k)=1/k;
    for j=1:N
        m(k,j) = j/(j+k-1)-1/(j+k);
    end
end
a=inv(m)*d;
y=ones(1,n);
for k=1:N
    y=y+a(k)*x.^k
end
```

exp_eq.m



Galerkin's method
Viscous Flow in Duct



Fluid Flow in Duct

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right\}$$

$$\frac{\partial p}{\partial z} = \text{const}$$

Poisson's Equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1$$

Test Functions

$$\tilde{w} = \sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N} a_{ij}\cos i\frac{\pi}{2}x\cos j\frac{\pi}{2}y$$

Test functions satisfy boundary conditions

Galerkin's Method
Viscous Flow in Duct

Remainder

$$R = -\left[\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N} a_{ij}\cos i\frac{\pi}{2}x\cos j\frac{\pi}{2}y\left\{\left(i\frac{\pi}{2}\right)^2+\left(j\frac{\pi}{2}\right)^2\right\}-1\right]$$

Inner product

$$\left(R,\cos k\frac{\pi}{2}x\cos \ell\frac{\pi}{2}y\right)\ ,\ i,j=1,3,5,\ldots$$

Analytical Integration

$$a_{ij}=\left(\frac{8}{\pi^2}\right)^2\frac{(-1)^{(i+j)/2-1}}{ij(i^2+j^2)}$$

Galerkin Solution

$$\tilde{w}=\left(\frac{8}{\pi^2}\right)^2\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N}\frac{(-1)^{(i+j)/2-1}}{ij(i^2+j^2)}\cos i\frac{\pi}{2}x\cos j\frac{\pi}{2}y$$

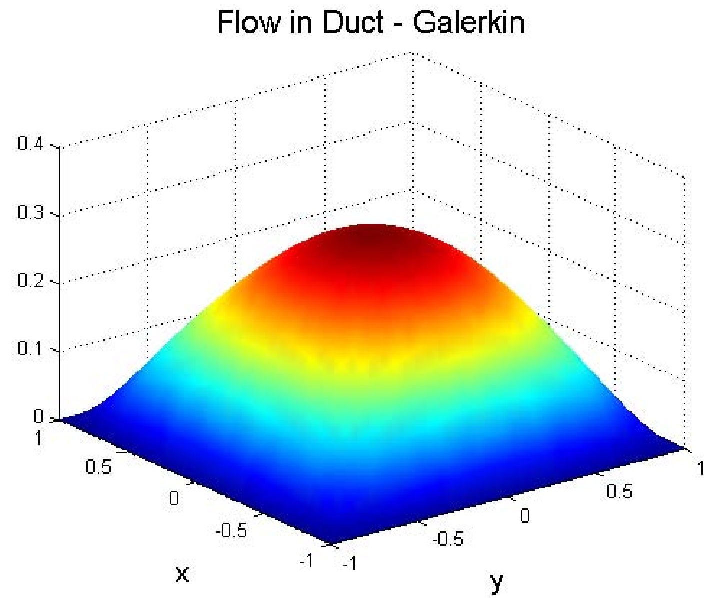
Flow Rate

$$\dot{q} = \int_{-1}^{1}\int_{-1}^{1}\tilde{w}(x,y)dxdy$$

$$= 2\left(\frac{8}{\pi^2}\right)^3\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N}\frac{1}{i^2j^2(i^2+j^2)}$$

duct_galerkin.m

```
x=[-1:h:1]';
y=[-1:h:1];
n=length(x); m=length(y); u=zeros(n,m);
Nt=5;
for j=1:n
 xx(:,j)=x; yy(j,:)=y;
end
for i=1:2:Nt
    for j=1:2:Nt
        u=u+(8/pi^2)^2*
          (-1)^((i+j)/2-1)/(i*j*(i^2+j^2))
          *cos(i*pi/2*xx).*cos(j*pi/2*yy);
    end
end
```



Computational Galerkin Methods

Differential Equation

$$L(u) = 0$$

Residuals

$$\begin{aligned} L(\tilde{u}) &= R \\ I(\tilde{u}) &= R_I \\ S(\tilde{u}) &= R_B \end{aligned}$$

$$\begin{aligned} R &= 0 \\ R_B &= 0 \end{aligned}$$

$$R, R_B \neq 0$$

Global Test Function

$$\tilde{u}(\mathbf{x},t) = u_0(\mathbf{x},t) + \sum_{j=1}^{N} a_j \phi(\mathbf{x},t)$$

Time Marching

$$\tilde{u}(\mathbf{x},t) = u_0(\mathbf{x},t) + \sum_{j=1}^{N} a_j(t) \phi(\mathbf{x})$$

Weighted Residuals

$$(R, w_k(\mathbf{x})) = 0, k = 1, \ldots N$$

$$\lim_{N\to\infty} ||\tilde{u} - u||_2 = 0$$

Boundary problem ($R = 0$)

- PDE satisfied exactly
- Boundary Element Method
  - Panel Method
  - Spectral Methods

Inner problem ($R_B = 0$)

- Boundary conditions satisfied exactly
- Finite Element Method
- Spectral Methods

Mixed Problem ($R, R_B \neq 0$)

Method of Weighted Residuals

Inner Product

$$(L(u), w) = 0$$

Discrete Form

$$(f, g) = \sum_{i=1}^{N} f_i g_i$$

Domain Method

$$w_k = \begin{cases} 1 & \text{in } D_k \\ 0 & \text{outside } D_k \end{cases}$$

Collocation

$$w_k(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_k)$$

$$R(\mathbf{x}_k) = 0$$

Least Squares

$$w_k = \frac{\partial R}{\partial a_k}$$

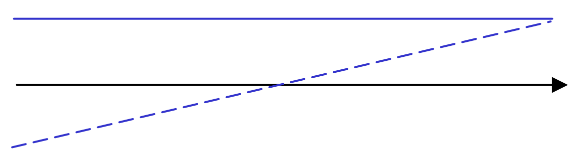
$$(\mathrm{R,R}) = \text{minimum}$$

Method of Moments

$$w_k(\mathbf{x}) = x^k \ , \quad k = 0, 1, \ldots, N$$

Galerkin

$$w_k(\mathbf{x}) = \phi_k(\mathbf{x})$$



Weighted Residuals

$$\frac{dy}{dx} - y = 0$$

$$\tilde{y} = 1 + \sum_{j=1}^{N} a_j x^j$$

Least Squares

$$\begin{bmatrix} 1/3 & 1/4 & 1/5 \\ 1/4 & 8/15 & 2/3 \\ 1/5 & 2/3 & 33/35 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2/3 \\ 3/4 \end{bmatrix}$$

Subdomain Method

$$\begin{bmatrix} 5/18 & 8/81 & 11/324 \\ 3/18 & 20/81 & 69/324 \\ 1/18 & 26/81 & 163/324 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Galerkin

$$\begin{bmatrix} 1/2 & 2/3 & 3/4 \\ 1/6 & 5/12 & 11/20 \\ 1/12 & 3/10 & 13/30 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$

Collocation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.75 & 0.625 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Methods of Weighted Residuals

Comparison of coefficients for approximate solution of $dy/dx - y = 0$

Scheme \ Coefficient	a_1	a_2	a_3
Least squares	1.0131	0.4255	0.2797
Galerkin	1.0141	0.4225	0.2817
Subdomain	1.0156	0.4219	0.2813
Collocation	1.0000	0.4286	0.2857
Taylor series	1.0000	0.5000	0.1667
Optimal $L_{2,d}$	1.0138	0.4264	0.2781

Figure by MIT OCW.

Comparison of approximate solutions of $dy/dx - y = 0$

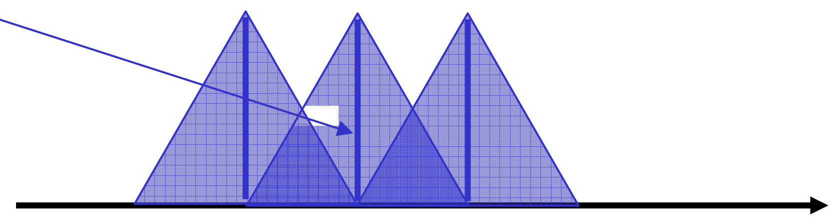
x	Least squares	Galerkin	Subdomain	Collocation	Taylor series	Optimal $L_{2,d}$	Exact
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.2219	1.2220	1.2223	1.2194	1.2213	1.2220	1.2214
0.4	1.4912	1.4913	1.4917	1.4869	1.4907	1.4915	1.4918
0.6	1.8214	1.8214	1.8220	1.8160	1.8160	1.8219	1.8221
0.8	2.2260	2.2259	2.2265	2.2206	2.2053	2.2263	2.2255
1.0	2.7183	2.7183	2.7187	2.7143	2.6667	2.7183	2.7183
$\|y_a - y\|_{2,d}$	0.00105	0.00103	0.00127	0.0094	0.0512	0.00101	

Figure by MIT OCW.

Solution for Nodal Unknowns

$$u(x,y) = \sum_{j=1}^{N} \overline{u}_j \phi_j(x,y)$$

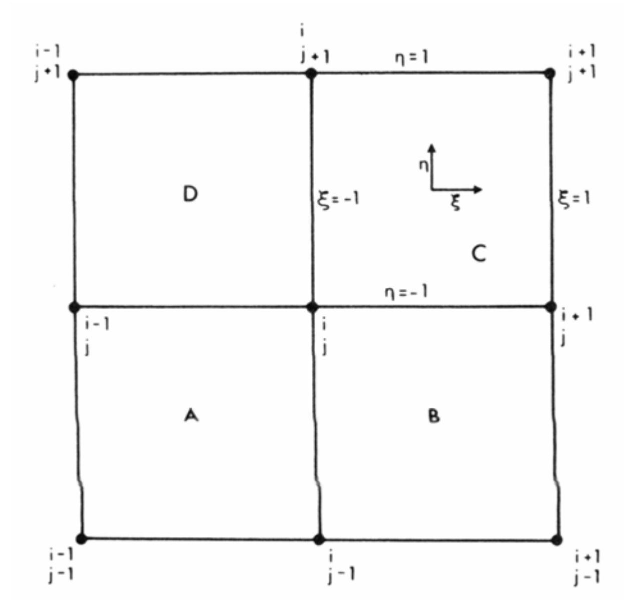
1 Dimension



$$u = \sum_{\ell=1}^{N} a_\ell \psi_\ell(x,y)$$

$$\Psi \mathbf{a} = \mathbf{u}$$

2 Dimensions



$$\mathbf{a} = \Psi^{-1}\mathbf{u}$$

$$\begin{aligned} u &= \sum_{\ell=1}^{N}\sum_{j=1}^{N} \Psi^{-1}_{\ell j}\overline{u}_j \psi_\ell(x,y) \\ &= \sum_{j=1}^{N} \overline{u}_j \left\{ \sum_{\ell=1}^{N} \Psi^{-1}_{\ell j} \psi_\ell(x,y) \right\} \end{aligned}$$

$$\phi_j(x,y) = \sum_{\ell=1}^{N} \Psi^{-1}_{\ell j} \psi_\ell(x,y)$$

Complex Boundaries
Isoparametric Elements

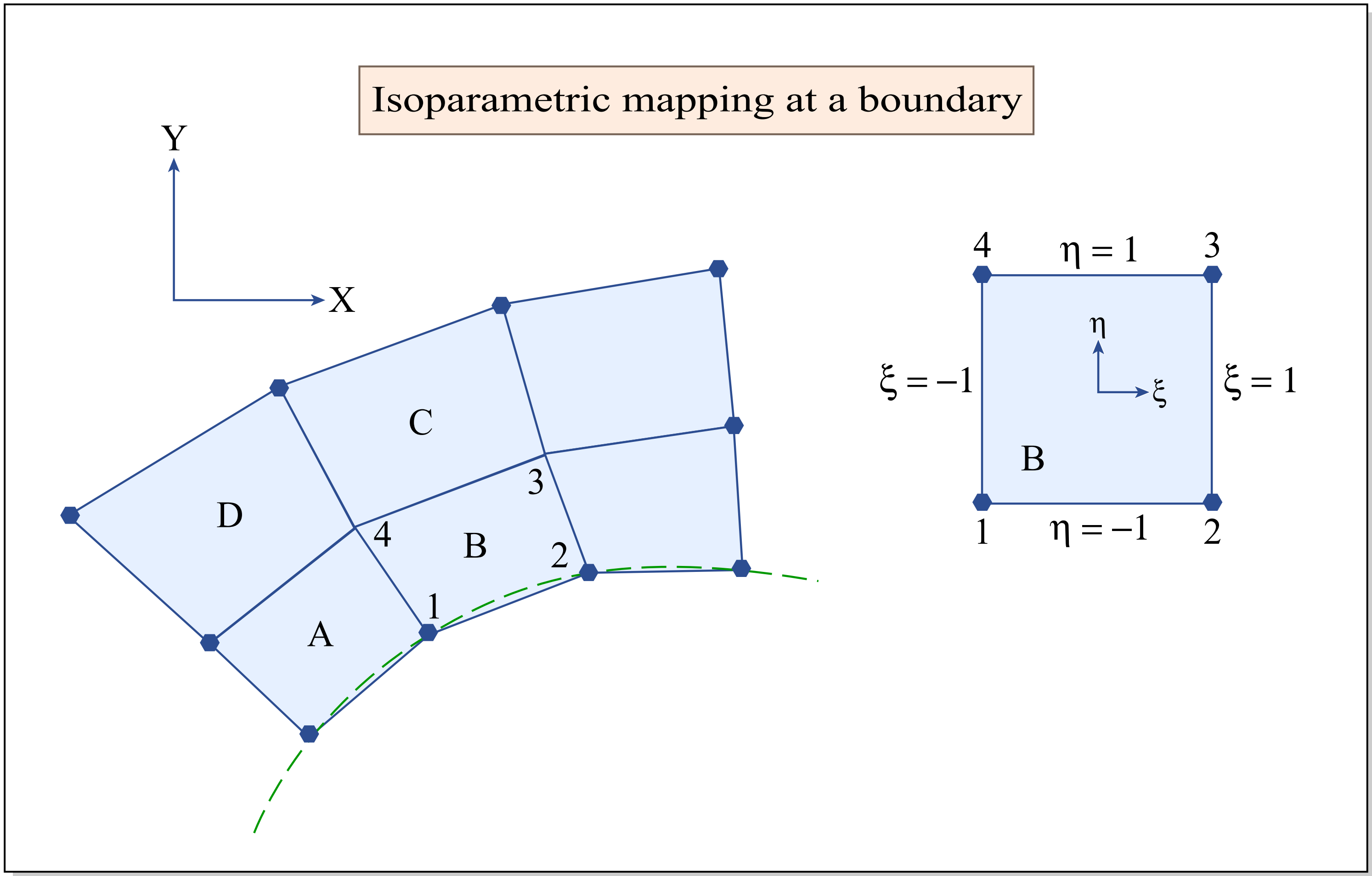


Figure by MIT OCW.

Finite Elements
1-dimensional Elements

Trial Function Solution

$$\tilde{u} = \sum_{j=1}^{N} N_j(x)\overline{u}_j$$

Interpolation Functions

$$N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$N_2 = \frac{x - x_3}{x_2 - x_3}$$

$$N_3 = \frac{x - x_2}{x_3 - x_2}$$

$$N_2 = \frac{x - x_4}{x_3 - x_4}$$

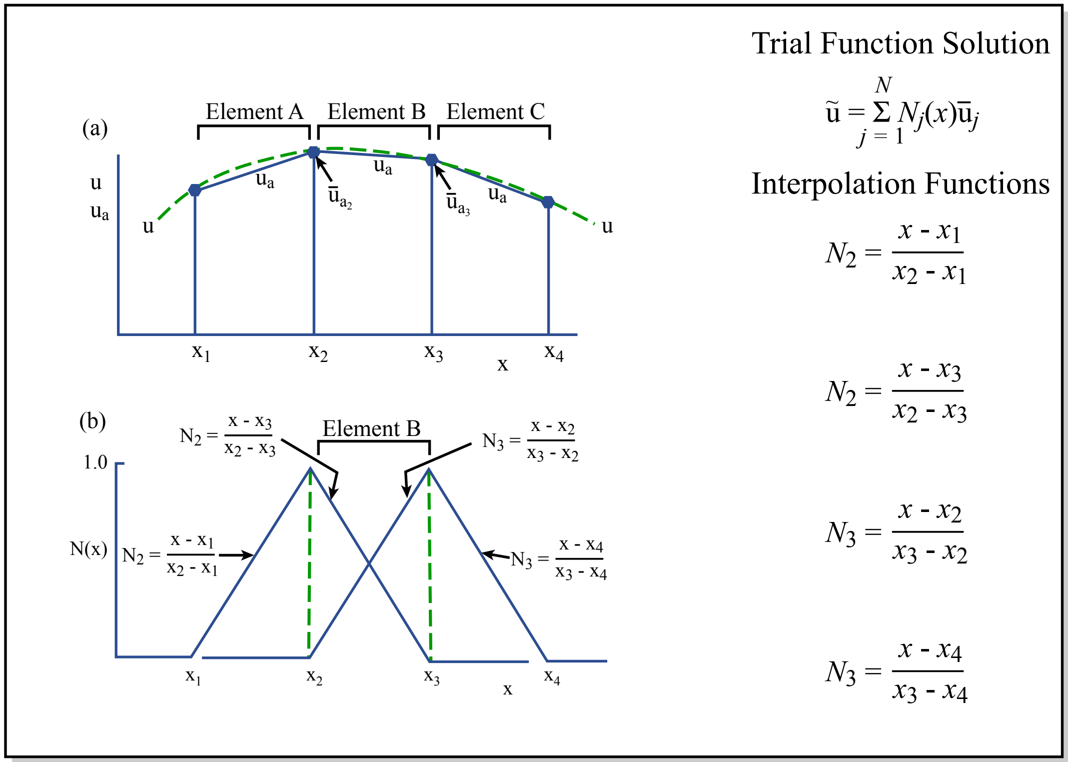


Figure by MIT OCW.

Finite Elements
1-dimensional Elements

Quadratic Interpolation Functions

$$N_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

$$N_3 = \frac{(x - x_4)(x - x_5)}{(x_3 - x_4)(x_3 - x_5)}$$

$$N_2 = \frac{(x - x_3)(x - x_5)}{(x_4 - x_3)(x_4 - x_5)}$$

$$N_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

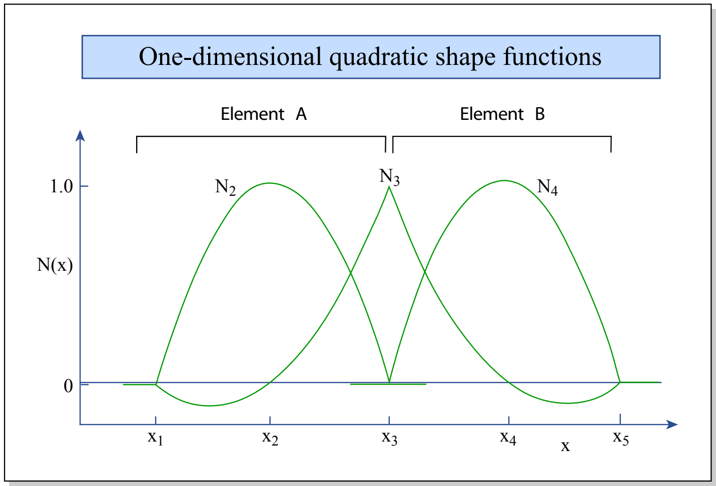


Figure by MIT OCW.

Finite Elements
2-dimensional Elements

$$\tilde{u} = \sum_{i=1}^{N}\sum_{j=1}^{N} N_{ij}(x)\overline{u}_{ij}$$

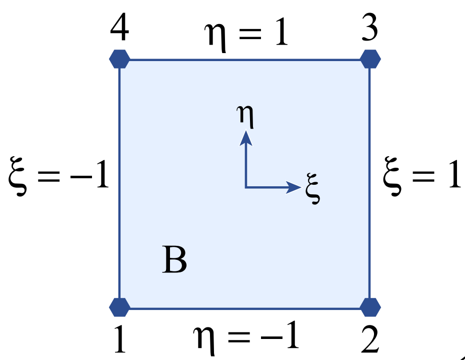


Figure by MIT OCW.

$$\tilde{u} = \sum_{\ell=1}^{4} N_\ell(\xi,\eta)\overline{u}_\ell$$

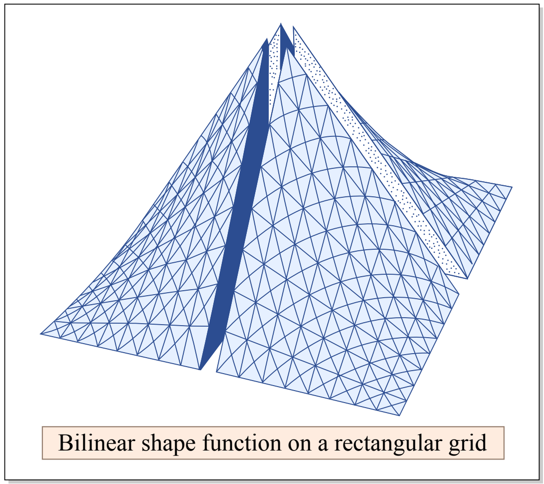


Figure by MIT OCW.

Linear Interpolation Functions

$$\begin{aligned}
N_1 &= 0.25(1-\xi)(1-\eta)\\
N_2 &= 0.25(1+\xi)(1-\eta)\\
N_3 &= 0.25(1+\xi)(1+\eta)\\
N_4 &= 0.25(1+\xi)(1+\eta)
\end{aligned}$$

$$N_\ell = 0.25(1+\xi_\ell\xi)(1+\eta_\ell\eta)$$

Quadratic Interpolation Functions

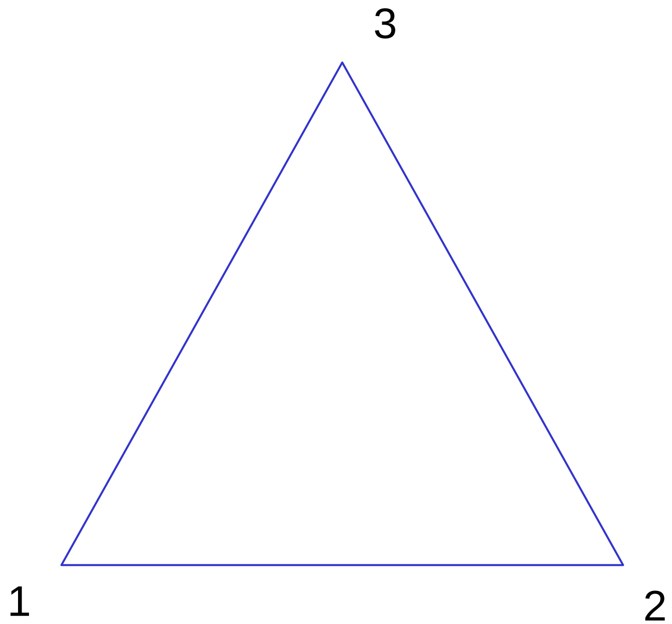
$$\prod_{r\neq i}\frac{(\xi-\xi_r)(\eta-\eta_r)}{(\xi_i-\xi_r)(\eta_i-\eta_r)}$$

$$\begin{aligned}
N_i &= 0.25\xi_i\xi(1+\xi_i\xi)\eta_i\eta(1+\eta_i\eta)\\
N_i &= 0.5(1-\xi^2)\eta_i\eta(1+\eta_i\eta)\ ,\ \xi_i = 0\\
N_i &= 0.5(1-\eta^2)\xi_i\xi(1+\xi_i\xi)\ ,\ \eta_i = 0\\
N_i &= (1-\xi^2)(1-\eta^2)
\end{aligned}$$

Finite Elements
2-dimensional Triangular Elements

Triangular Coordinates

$$
\begin{aligned}
L_1 &= \frac{a_1 + b_1 x + c_1 y}{2A_T} \\
L_2 &= \frac{a_2 + b_2 x + c_2 y}{2A_T} \\
L_3 &= 1 - N_1 - N_2
\end{aligned}
$$



$$
\begin{aligned}
a_1 &= x_2 y_3 - x_3 y_2 \\
b_1 &= y_2 - y_3 \\
c_1 &= x_3 - x_2
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= x_3 y_1 - x_1 y_3 \\
b_2 &= y_3 - y_1 \\
c_2 &= x_1 - x_3
\end{aligned}
$$

Interpolation Functions

$$
\begin{aligned}
N_1 &= L_1 \\
N_2 &= L_2 \\
N_3 &= L_3
\end{aligned}
$$

Two-Dimensional Finite Elements
Flow in Duct

Finite Element Solution

$$\tilde{w} = \sum_{j=1}^{N} \overline{w}_j N_j(x,y)$$

$$N_j = 0.25(1+\xi_j\xi)(1+\eta_j\eta)$$

$$\left(\frac{\partial^2 \tilde{w}}{\partial x^2}, N_k\right) + \left(\frac{\partial^2 \tilde{w}}{\partial x^2}, N_k\right) = (-1, N_k)$$

Integration by Parts

$$\left(\frac{\partial^2 w}{\partial x^2}, N_k\right) \equiv \int_{-1}^{1} \frac{\partial^2 w}{\partial x^2} N_k = \left[\frac{\partial w}{\partial x} N_k\right]_{-1}^{1} - \int_{-1}^{1} \frac{\partial w}{\partial x}\frac{dN_k}{dx}$$

$$\left(\frac{\partial^2 \tilde{w}}{\partial x^2}, N_k\right) = -\left(\frac{\partial \tilde{w}}{\partial x}, \frac{\partial N_k}{\partial x}\right)$$

Algebraic Equations

$$-\sum_{j=1}^{N}\left(\int_{-1}^{1}\int_{-1}^{1} \frac{\partial N_j}{\partial x}\frac{\partial N_k}{\partial x} + \frac{\partial N_j}{\partial y}\frac{\partial N_k}{\partial y} \;\; dxdy\right)\overline{w}_j = -\int_{-1}^{1}\int_{-1}^{1} 1 N_k dxdy, k = 1,\ldots N$$