– SOLUTIONS –

$\boxed{1}$ PARSEVAL'S THEOREM

Show:
$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(u)|^2 du$$

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} f(x) \cdot f^*(x)\, dx =$$

$$= \int_{-\infty}^{+\infty} f(x) \cdot \left[\int_{-\infty}^{+\infty} F^*(u)\, e^{-i 2\pi u x} du \right] dx$$

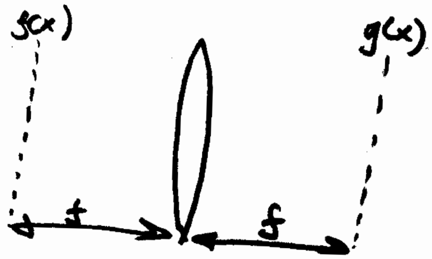
Reversing the order of integration we get:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} F^*(u) \left[\int_{-\infty}^{+\infty} f(x)\, e^{-i2\pi x u} dx \right] du.$$

$$= \int_{-\infty}^{+\infty} F^*(u)\, F(u)\, du = \int_{-\infty}^{+\infty} |F(u)|^2 du \quad //$$

Q.E.D.

(b) Take the following case:



We know that the relation between the input field and the output field is:

$$g(x) = G \cdot \mathcal{F}\{f(x)\} \qquad G: \text{Complex Constant}.$$

The intensity is then: $|g(x)|^2 = |\mathcal{F}\{f(x)\}|^2$

The total energy in the input beam is:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx.$$

And the total energy in the output beam is:

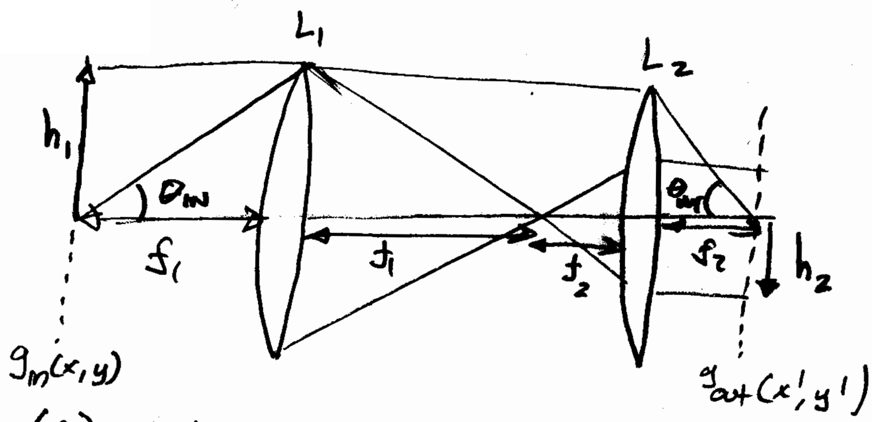
$$\int_{-\infty}^{+\infty} \Big| \underbrace{\mathcal{F}\{f(x)\}}_{F(u)} \Big|^2 du$$

Hence,
$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(u)|^2 du.$$

$\Rightarrow$ This relation implies that energy is conserved in the case of a lossless optical system (like the ones we've modelled)

Telescopes & Magnifications.

$f_1 > f_2.$



(a) Lateral magnifier or demagnifier?

$$g_{out}(x',y') \simeq g\left(-x'\frac{f_1}{f_2}, -y'\frac{f_1}{f_2}\right)$$

Given that $\frac{f_1}{f_2} > 1 \longrightarrow$ when $x = h_1 \rightarrow h_1 = -h_2'\frac{f_1}{f_2}.$

$$\Rightarrow h_2' = -\underbrace{\left(\frac{f_2}{f_1}\right)}_{<1} h_1$$

$\Rightarrow$ The system is operating as a demagnifier in the lateral coordinate.

(b) Angular magnifier?

From the input: $\theta_{IN} = \tan^{-1}\left(\frac{h_1}{f_1}\right)$.

From the output: $\theta_{OUT} = \tan^{-1}\left(\frac{h_1}{f_2}\right)$

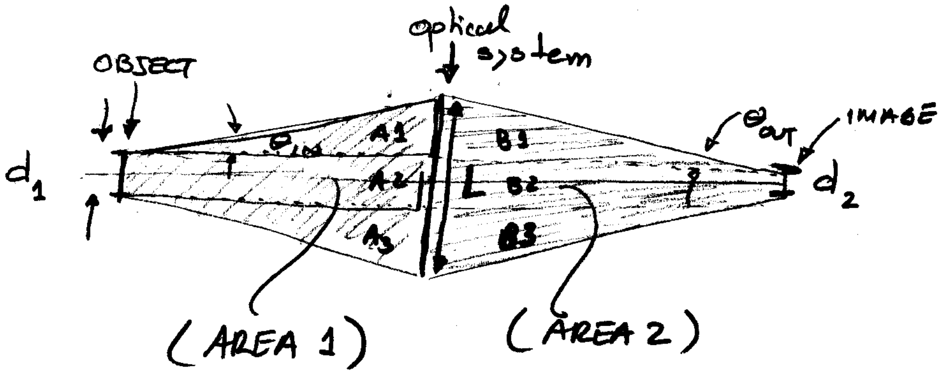
$$\frac{\alpha_{OUT}}{\alpha_{IN}} = \frac{\tan^{-1}\left(\frac{h_1}{f_2}\right)}{\tan^{-1}\left(\frac{h_1}{f_1}\right)} > 1 \quad \text{if } f_1 > f_2$$

$\Rightarrow$ System is magnifying angularly.

(c) Consistency.

The reason a 4F system magnifies angularly and not laterally has to do with the fact that energy has to be conserved.

To illustrate the case consider the following diagram:



Cross Section.

Energy conservation constraint $\Rightarrow$ AREA 1 = AREA 2.

$$\text{AREA 1} = A1 + A2 + A3$$

$$= \frac{1}{2}\cdot\left(\frac{L-d_1}{2}\right)\cdot\left(\frac{L-d_1}{2\tan\theta_{IN}}\right) + d_1\cdot\frac{L-d_1}{2\tan\theta_{IN}} + \frac{1}{2}\left(\frac{L-d_1}{2}\right)\left(\frac{L-d_1}{2\tan\theta_{IN}}\right)$$

$$= \frac{(L-d_1)^2}{4\tan\theta_{IN}} + d_1\cdot\frac{L-d_1}{2\tan\theta_{IN}} = \frac{1}{2\tan\theta_{IN}}\left(\frac{1}{2}(L^2+d_1^2-2Ld_1) + d_1 L - d_1^2\right)$$

$$= \frac{1}{2\tan\theta_{IN}}\left(\frac{1}{2}(L^2-d_1^2)\right) = \frac{L^2-d_1^2}{4\tan\theta_{IN}}$$

$$\text{AREA 2} = B1 + B2 + B3 = \frac{L^2-d_2^2}{4\tan\theta_{OUT}}$$

conserv. of Energy $\Rightarrow$ $\dfrac{L^2-d_1^2}{\cancel{4}\tan\theta_{IN}} = \dfrac{L^2-d_2^2}{\cancel{4}\tan\theta_{OUT}} \Rightarrow \dfrac{\tan\theta_{OUT}}{\tan\theta_{IN}} = \dfrac{L^2-d_2^2}{L^2-d_1^2}$

$$\Rightarrow \approx \frac{\theta_{OUT}}{\theta_{IN}} = \underset{\text{some constant}}{K}\cdot\left(\frac{d_2}{d_1}\right)^2 \Rightarrow \underbrace{\frac{\theta_{OUT}}{\theta_{IN}}}_{\text{Ang. mag.}}\left(\underbrace{\frac{d_1}{d_2}}_{\text{lateral magnification}}\right)^2 = \text{Constant}$$

$\Rightarrow$ Direct tradeoff between angular magnification & lateral magnification

(a) Is the Fourier Transform Linear?

Recall linearity implies:

$$f_1(x) \longrightarrow \boxed{\mathcal{S}'} \longrightarrow F(u)$$

$$g(x) \longrightarrow \boxed{\mathcal{S}'} \longrightarrow G(u)$$

then $\alpha f(x) + \beta g(x) \longrightarrow \boxed{\mathcal{S}'} \longrightarrow \alpha \tilde{F}(u) + \beta G(u)$

Apply to Fourier Transform:

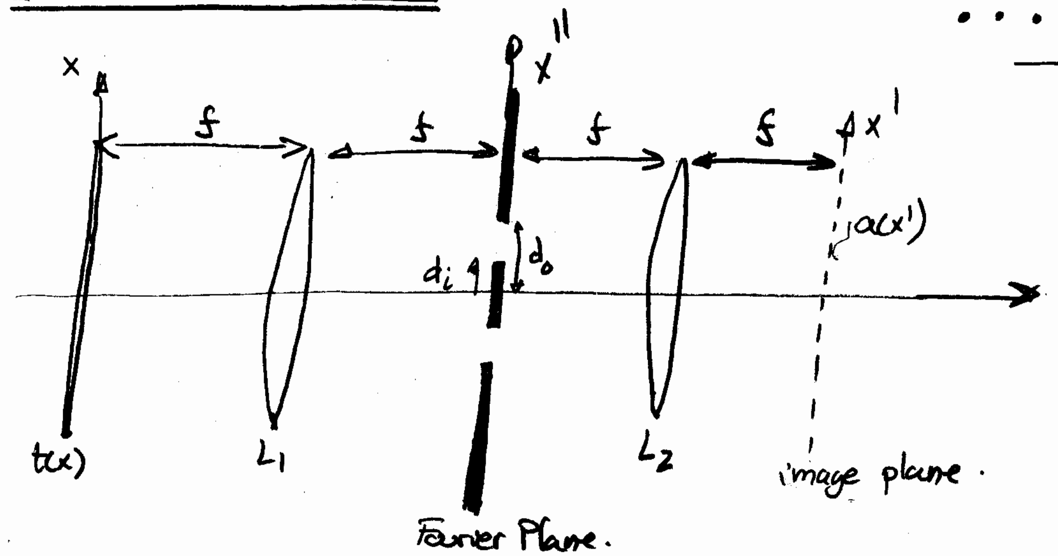
$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \int_{-\infty}^{+\infty} (\alpha f(x) + \beta g(x))\, e^{-j2\pi u x}\, dx$$

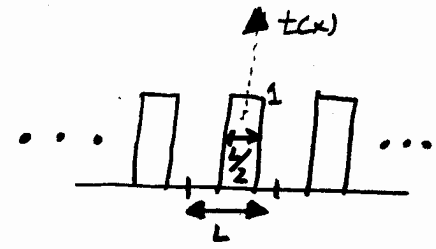
$$= \int_{-\infty}^{+\infty} \alpha f(x) e^{-j2\pi u x} dx + \int_{-\infty}^{+\infty} \beta\, g(x) e^{-j2\pi u x} dx = \underline{\underline{\alpha F(u) + \beta G(u)}}.$$

LINEAR SYSTEM.

(b) Given the fact that the Fourier Transform is not shift invariant, we can not define a transfer function.

Spatial Filtering.

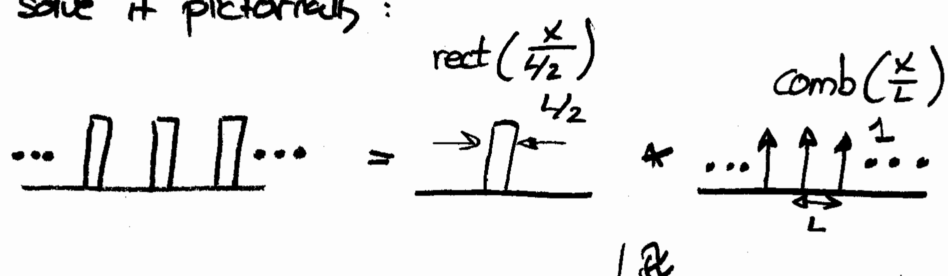




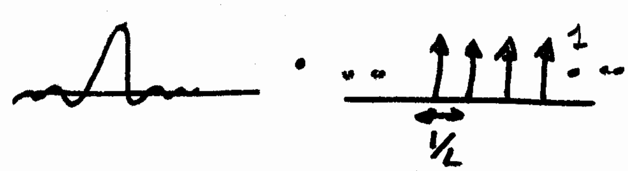
$f = 10$ cm
$L = 20\ \mu m$
$\lambda = 0.5\ \mu m$
$d_i = 1.5$ mm
$d_o = 8.5$ mm

First lets solve it pictorially:

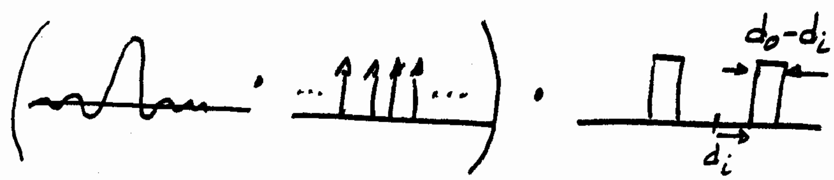
INPUT: t(x):



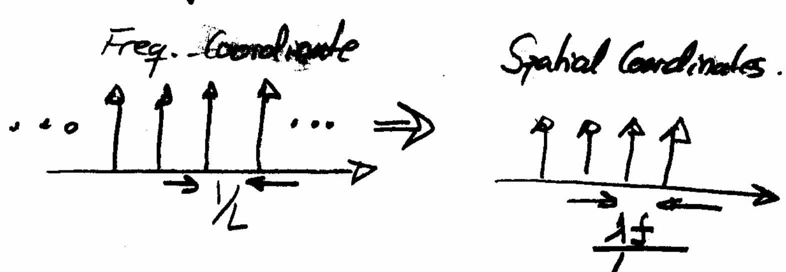
L_1 performs a Fourier Transform of t(x):

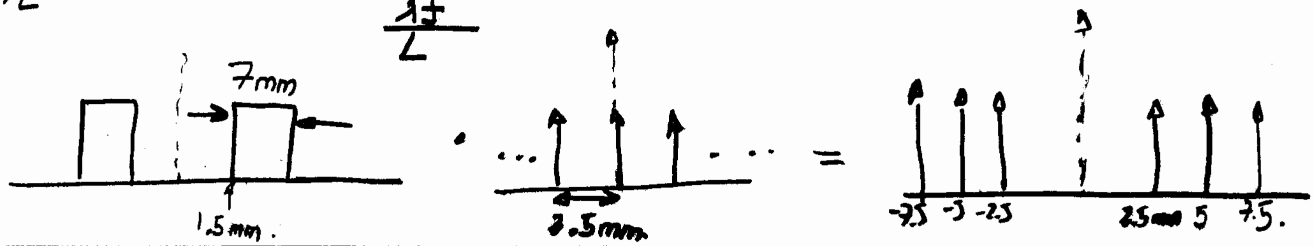


It then gets filtered by an aperture @ the Fourier Plane:

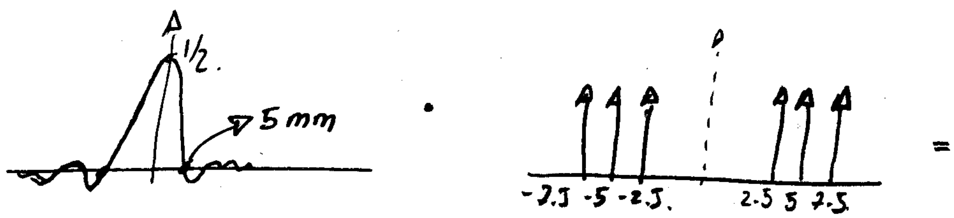


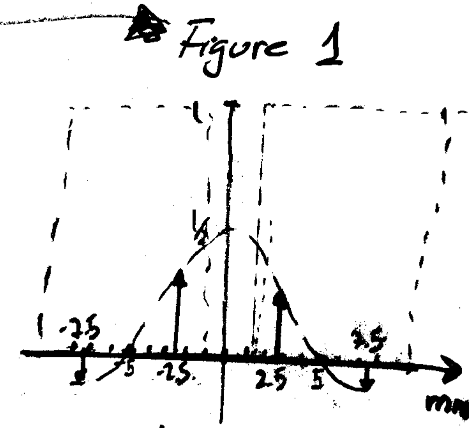
Only a finite number of impulses will be able to make it through the filter. Lets calculate how many.





Then, the field after the filter will be:





only two impulses remain from the product.

Lets see what this product gives analytically:

The input transparency is given by:

(1) $t(x) = \text{comb}\left(\frac{x}{L}\right) * \text{rect}\left(\frac{x}{L/2}\right)$

The field right before the filter:

(2) $T(u) = \frac{1}{L}\text{comb}(Lu) \cdot \frac{L}{2}\,\text{sinc}\left(\frac{L}{2}u\right) = \frac{1}{2}\cdot\text{comb}(Lu)\,\text{sinc}\left(\frac{L}{2}u\right)$

The field right after the filter will be:

(3) $T\left(\frac{x''}{\lambda f}\right) = \frac{1}{2}\cdot\text{comb}\left(\frac{L}{\lambda f}x''\right)\underbrace{\text{sinc}\left(\frac{L}{2\lambda f}x''\right)} \cdot \left[\text{rect}\left(\frac{x'' - \frac{d_o - d_i}{2}}{d_o - d_i}\right) + \text{rect}\left(\frac{x'' + \frac{d_o - d_i}{2}}{d_o - d_i}\right)\right]$

$\left(\frac{\lambda}{Lf} = 400\ m^{-1}\right)$ $\quad\text{sinc}(200x'') \Rightarrow$ zero at $x'' = 5$ mm $\quad\left(\frac{1}{200m} = 5\text{ mm}\right)$

$$T\left(\frac{x''}{\lambda f}\right) = \Big(\big(\delta(x''-2.5) + \delta(x''+2.5)\big) + \big(\delta(x''-5) + \delta(x''+5)\big) + \big(\delta(x''-7.5) + \delta(x''+7.5)\big)\Big) \cdot \frac{1}{2}\,\text{sinc}(0.2x'')$$

w/ x'' in <u>mm</u>.

From figure 1 we know that only 4 impulses survive this product:

$$T\left(\frac{x''}{\lambda f}\right) = \frac{1}{2}\,\text{sinc}(0.2x'')\cdot\left[\delta(x''-2.5) + \delta(x''+2.5) + \delta(x''-7.5) + \delta(x''+7.5)\right].$$

Then, L_2 performs the Fourier transform of the field after the filter.

This will simply be the Fourier Transform of 2 pairs of scaled impulses.

$$a(x') = \tilde{\mathcal{F}}\left\{ \frac{1}{2}\operatorname{sinc}(0.2x'')\left[\delta(x''-2.5) + \delta(x''+2.5) + \delta(x''-7.5) + \delta(x''+7.5) \right]\right.$$

$$= \tilde{\mathcal{F}}\left\{\left(\underbrace{\frac{1}{2}\operatorname{sinc}(0.2\cdot(2.5))}_{a}\cdot\left(\delta(x''-2.5)+\delta(x''+2.5)\right) + \underbrace{\frac{1}{2}\operatorname{sinc}(0.2\cdot(7.5))}_{b}\left(\delta(x''-7.5)+\delta(x''+7.5)\right)\right)\right]$$

$$= \tilde{\mathcal{F}}\left\{\left(a\underbrace{\left(\delta(x''-2.5)+\delta(x''+2.5)\right)}_{\substack{\tilde{\mathcal{F}} \\ \downarrow \\ \text{cosine}}} + b\underbrace{\left(\delta(x''-7.5)+\delta(x''+7.5)\right)}_{\substack{\tilde{\mathcal{F}} \\ \downarrow \\ \text{cosine.}}}\right)\right].$$

$$a(x') = a\cos\left((2.5\times10^{-3})\cdot 2\pi\cdot\frac{x'}{\lambda f}\right) + b\cos\left((7.5\times10^{-3})\cdot 2\pi\cdot\frac{x'}{\lambda f}\right)$$

$$= \underline{\underline{a\cos\left(1\times10^{5}\cdot\pi\cdot x'\right) + b\cos\left(3\times10^{5}\cdot\pi\cdot x'\right)}}$$