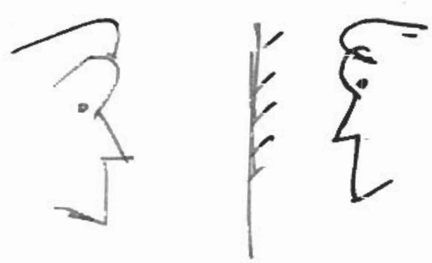
PS#2

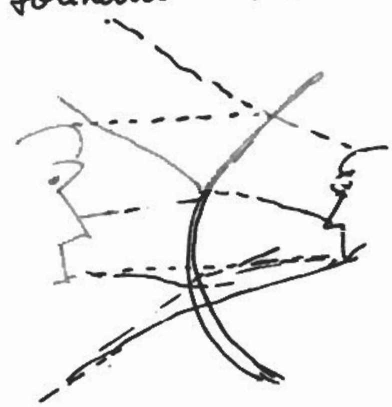
Solutions

1) Image on spoon.

Most people got this problem only partially right. Although most of you did recognize the fact that the image formed would be demagnified and erect, ~~so~~ there is another additional subtle point; The image that you see would also be distorted. The demagnification would not be uniform and you would see your forehead and chin demagnified more. (Try it!)

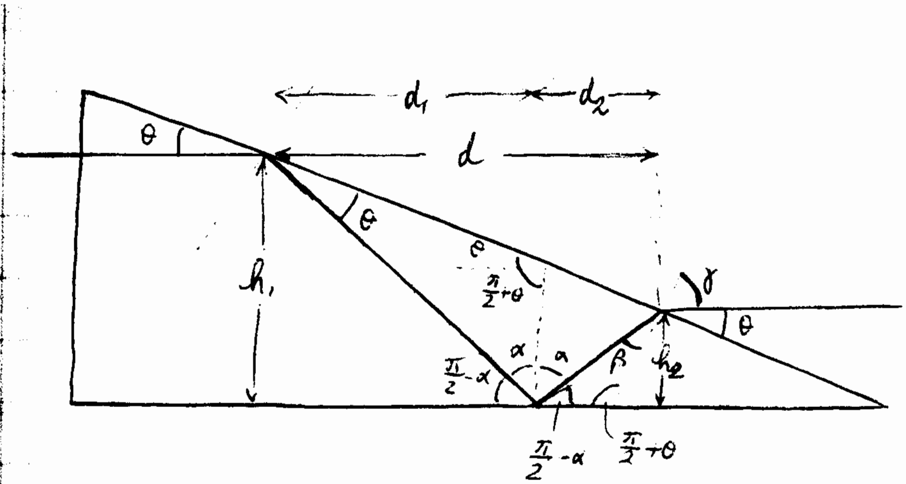


plane mirror



Concave mirror

Handedness changes!



$$\theta + \left(\frac{\pi}{2}+\theta\right) + \alpha = \pi \Rightarrow \alpha = \frac{\pi}{2} - 2\theta$$

$$\left(\frac{\pi}{2}-\alpha\right) + \left(\frac{\pi}{2}+\theta\right) + \beta = \pi \Rightarrow 2\theta - \frac{\pi}{2} + \theta + \beta = 0 \Rightarrow \beta = \frac{\pi}{2} - 3\theta$$

$$\gamma = \frac{\pi}{2} - \theta$$

$$n \sin\beta = \sin\gamma \Rightarrow n \sin\left(\frac{\pi}{2} - 3\theta\right) = \sin\left(\frac{\pi}{2}-\theta\right) \Rightarrow n\cos 3\theta = \cos\theta \Rightarrow$$

$$\Rightarrow n = \frac{\cos\theta}{\cos 3\theta}$$

Ensure TIR at reflections 1,2

(1) $n \sin\left(\frac{\pi}{2}-\theta\right) > 1 \Leftrightarrow n\cos\theta > 1 \Leftrightarrow \cos\theta > \frac{1}{n} \Leftrightarrow \theta < \arccos\frac{1}{n}$

(2) $n \sin\alpha > 1 \Leftrightarrow n\cos 2\theta > 1 \Leftrightarrow \theta < \frac{1}{2}\arccos\frac{1}{n}$

Exit condition: $n \sin\beta < 1 \Leftrightarrow n\cos 3\theta < 1 \Leftrightarrow \theta > \frac{1}{3}\arccos\frac{1}{n}$

$\Rightarrow$ for this to work, $\frac{1}{3}\arccos\frac{1}{n} < \theta < \frac{1}{2}\arccos\frac{1}{n}$

but $n = \frac{\cos\theta}{\cos 3\theta} \Rightarrow \begin{cases} \arccos\frac{1}{n} < 3\theta \Rightarrow \frac{1}{n} > \cos 3\theta \Rightarrow \frac{\cos 3\theta}{\cos\theta} > \cos 3\theta \Rightarrow \cos\theta < 1 \text{ ok} \\ \arccos\frac{1}{n} > 2\theta \Rightarrow \frac{\cos 3\theta}{\cos\theta} < \cos 2\theta \Rightarrow \cos 3\theta < \cos 2\theta \cos\theta \end{cases}$

true because $\cos 3\theta = \cos 2\theta\cos\theta - \underbrace{\sin 2\theta \sin\theta}_{\geq 0} \Rightarrow$ always consistent!

$$\tan\left(\frac{\pi}{2}-\alpha\right)=\frac{h_1}{d_1} \Rightarrow d_1=\frac{h_1}{\tan(\frac{\pi}{2}-\alpha)}=\frac{\cancel{h_1 \sin\alpha}}{\cancel{\cos\alpha}}=\frac{h_1}{\tan 2\theta}=$$

$$=\frac{h_1\cos 2\theta}{\sin 2\theta}$$

similarly $d_2=\frac{h_2\cos 2\theta}{\sin 2\theta} \Rightarrow d_1+d_2=(h_1+h_2)\frac{\cos 2\theta}{\sin 2\theta}$

but at the same time $d_1+d_2=d=\frac{h_1-h_2}{\tan\theta}$

$$\Rightarrow (h_1+h_2)\frac{\cos 2\theta}{\sin 2\theta}=\frac{h_1-h_2}{\tan\theta} \Rightarrow \frac{h_1+h_2}{h_1-h_2}=\frac{\sin 2\theta}{\cos 2\theta}\cdot\frac{\cos\theta}{\sin\theta}\equiv t$$

$$\Rightarrow \frac{1+f}{1-f}=t \Rightarrow 1+f=t-tf \Rightarrow 1-t=-(1+t)f \Rightarrow$$

$$\Rightarrow f=\frac{t-1}{t+1}=\frac{\frac{\sin 2\theta\cos\theta}{\cos 2\theta\sin\theta}-1}{\frac{\sin 2\theta\cos\theta}{\cos 2\theta\sin\theta}+1}=\frac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\sin 2\theta\cos\theta+\cos 2\theta\sin\theta}=$$

$$=\frac{\sin\theta}{\sin 3\theta}$$

but $n=\frac{\cos\theta}{\cos 3\theta} \Rightarrow \cos\theta=\frac{1}{2}\sqrt{3+\frac{1}{n}}$

<u>Proof.</u> $\cos 3\theta=\cos 2\theta\cos\theta-\sin 2\theta\sin\theta=$

$$=(\cos^2\theta-\sin^2\theta)\cos\theta-2\sin^2\theta\cos\theta=\cos^3\theta-3\sin^2\theta\cos\theta=$$

$$=\cos\theta\left[\cos^2\theta-3(1-\cos^2\theta)\right]=\cos\theta(4\cos^2\theta-3)$$

$$\Rightarrow n=\frac{\cos\theta}{\cos\theta(4\cos^2\theta-3)} \Rightarrow 4\cos^2\theta-3=\frac{1}{n} \Rightarrow \cos^2\theta=\frac{1}{4}\left(3+\frac{1}{n}\right)$$

ok

$$\sin^2\theta = 1 - \frac{1}{4}\left(3 + \frac{1}{n}\right) = \frac{1}{4}\left(1 - \frac{1}{n}\right)$$

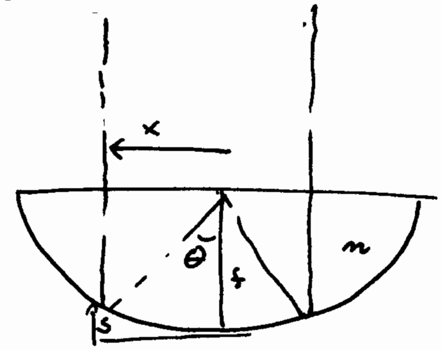
$$\sin 3\theta = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta = 2\sin\theta\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)\sin\theta =$$

$$= \sin\theta\left(3\cos^2\theta - \sin^2\theta\right) = \sin\theta\left[3\cos^2\theta - \left(1 - \cos^2\theta\right)\right] =$$

$$= \sin\theta\left(4\cos^2\theta - 1\right) \Rightarrow$$

$$\Rightarrow f = \frac{\sin\theta}{\sin\theta\left(4\cos^2\theta - 1\right)} = \frac{1}{3 + \frac{1}{n} - 1} = \frac{1}{2 + \frac{1}{n}} = \frac{n}{1+2n}$$

(3) Mirror in pool!



(a) Galint

Recognize that as x increases, θ increases also. Since $n > 1$, there is a possibility of TIR. We need to find this.

For TIR, $\sin\theta = 1/n$ — ①

also $\sin\theta = \dfrac{x}{\sqrt{(f-s)^2 + x^2}}$ — ②

From ① & ②

$n^2 x^2 = (f-s)^2 + x^2$

$\therefore\ x\sqrt{n^2-1} = f - s$ $\qquad s = x^2/4f$

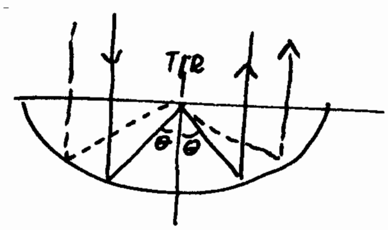
$x\sqrt{n^2-1} = f - s$

Solve to get

$x = 2f\left(n - \sqrt{n^2-1}\right)$

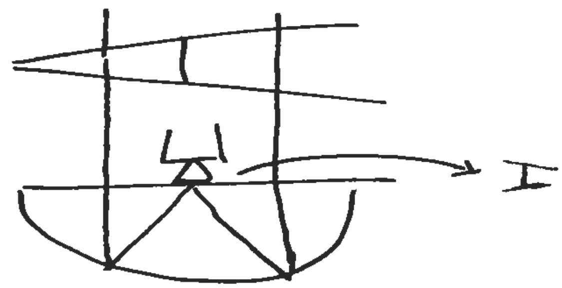
$\dfrac{x}{2f} = \left(n - \sqrt{n^2-1}\right)$

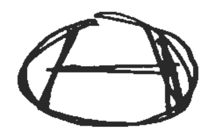
(b)



A parallel ray bundle would also come out parallel by symmetry.

(c)



The camera would see 

(A truncated A