Class 4 Outline

Experiment Design and Output Analysis:

1. **Theory and Construction of Confidence Intervals**
2. **Estimation and Prediction Errors**
3. **Non-Terminating Simulations (Steady State)**

Experimental Design Issues

Static Simulation (Monte-Carlo) →

- ***How many replications should be run ?***

Dynamic Simulation (Discrete-Event)

Terminating →

- ***How many replications should be run ?***

Non-terminating →

- ***How many replications should be run?***

For each replication:

- ***When to start collecting data?***
- ***How much data should be collected?***

What is the prediction error involved when using simulation results?

How Many Replications?

- **Simulation output: $Y_1, Y_2, Y_3, \ldots, Y_n$**
  We (typically) want to estimate E[Y]!

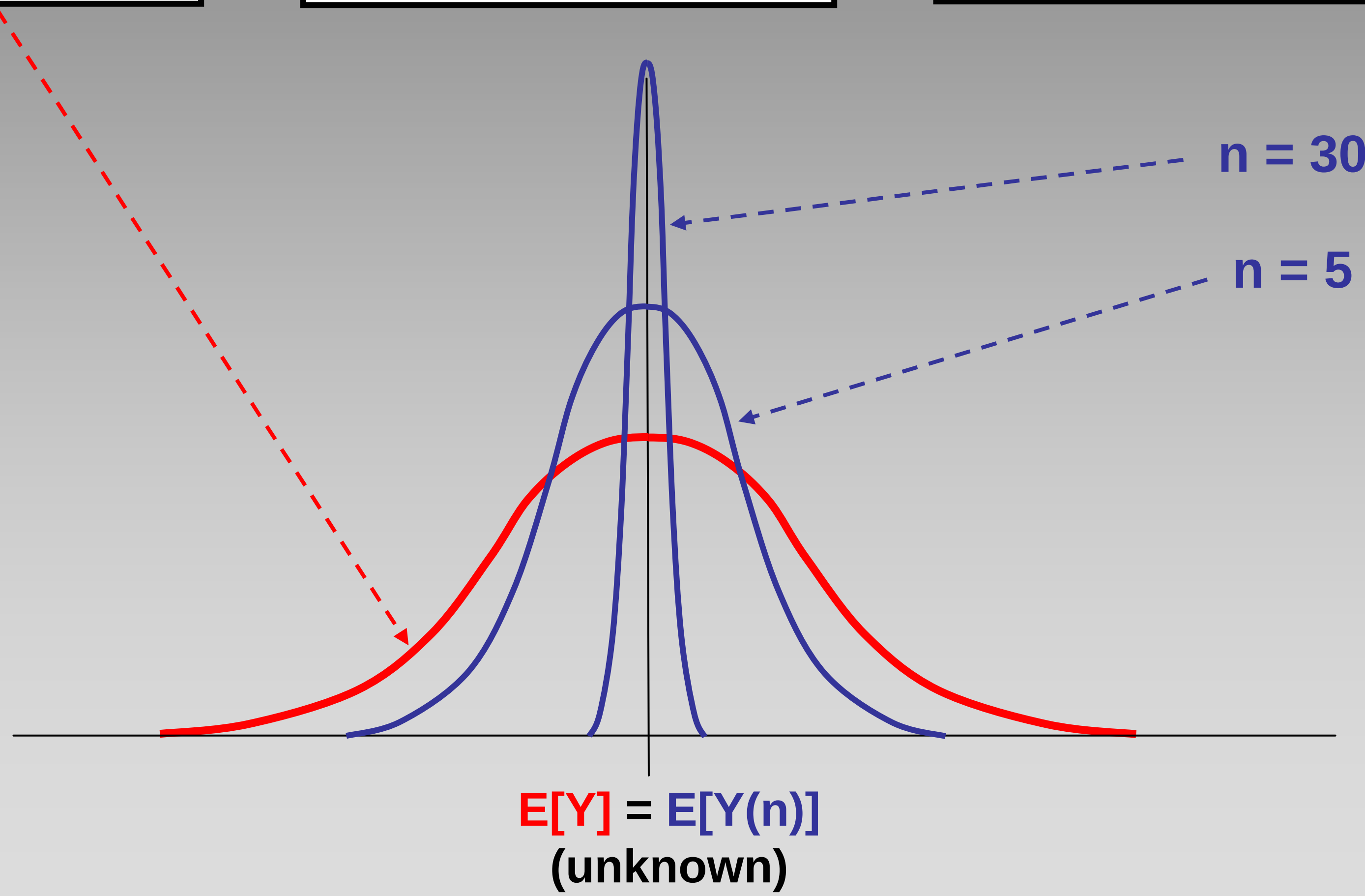
1. **What is the accuracy of the estimator:**

$$Y(n) = (Y_1+Y_2+Y_3+ \ldots +Y_n) / n \ ?$$

2. **How much should n be (number of independent replications) in order to guarantee a given estimation accuracy?**

Sample Mean Review

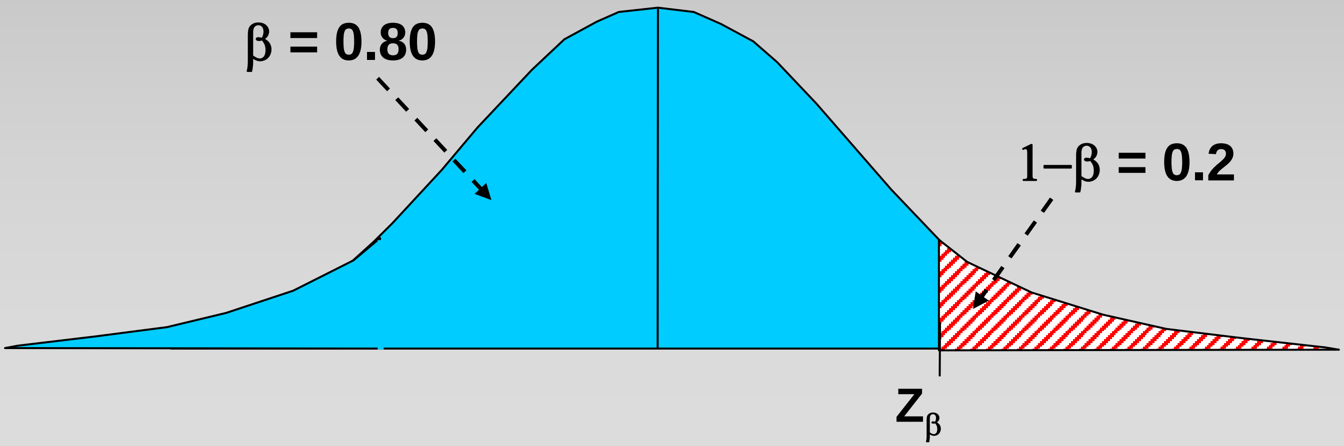
Unknown distribution Y → **Simulation Output:** $Y_1, Y_2, Y_3, \ldots, Y_n$ → **Sample Mean:** $Y(n) = (Y_1+Y_2+Y_3+ \ldots +Y_n\ ) / n$



What is a “Fractile”?

- **Let β be a number in (0,1) e.g. 0.80**
- **z_β is the β-th fractile of a standard normal distribution, and is defined as:**

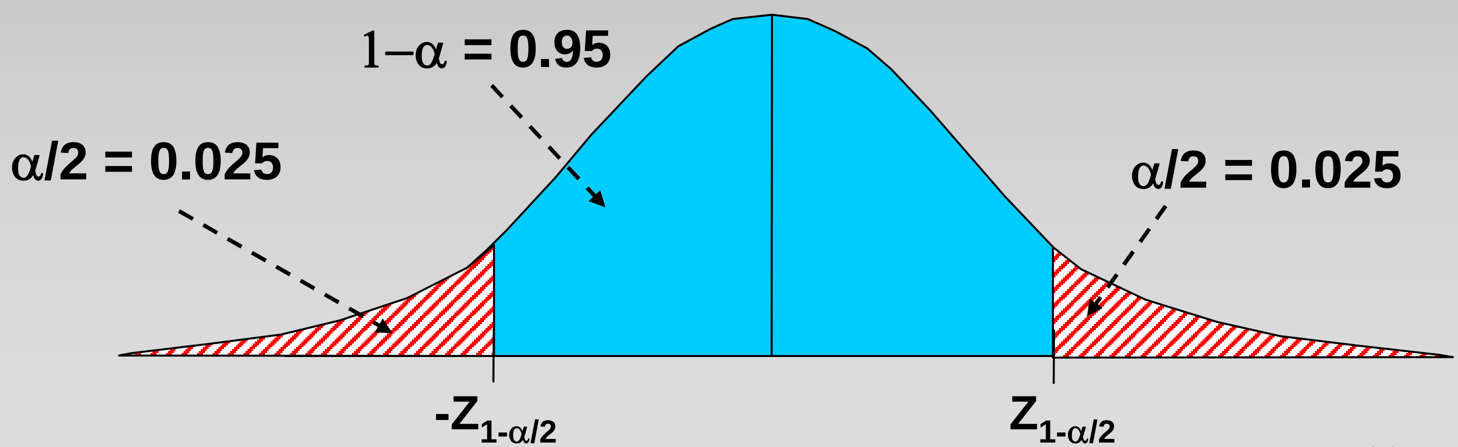
$$P(N(0,1) \leq z_\beta) = \beta$$



Two-Sided Tail

- **Let α be a number in (0,1) e.g. 0.05**
- **Let $z_{1-\alpha/2}$ the (1-α/2)-th fractile of a standard normal distribution, we have:**

$$P(-z_{1-\alpha/2} \leq N(0,1) \leq z_{1-\alpha/2}) = 1-\alpha$$



Statistical Estimation Theory

- **Define:**

$$\bar{Y}(n) = \frac{1}{n}\sum_{i=1}^{n} Y_i \qquad S^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}(n))^2$$

- **A version of the CLT says that when n → ∞ :**

$$\frac{\bar{Y}(n) - E[Y]}{\sqrt{\frac{S^2(n)}{n}}} \rightarrow N(0,1)$$

- **For smaller (n<30) values of n, a good approx. is:**

$$\frac{\bar{Y}(n) - E[Y]}{\sqrt{\frac{S^2(n)}{n}}} \rightarrow t_{n-1}$$

So What?

- **From the CLT:**

$$\frac{Y(n)-E[Y]}{\sqrt{\frac{S^2(n)}{n}}} \rightarrow N(0,1)$$

- **So that when n is large, we can write:**

$$P\left(-z_{1-\alpha/2} \leq \frac{Y(n)-E[Y]}{\sqrt{\frac{S^2(n)}{n}}} \leq z_{1-\alpha/2}\right) = 1-\alpha$$

- **Re-arranging gives:**

$$P\left(Y(n) - z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \leq E[Y] \leq Y(n) + z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}\right) = 1-\alpha$$

- **This is the definition of a (1-α)% confidence interval!!!**

Building Confidence Intervals

- **For n large (n>30), the (1- α)% confidence interval is:**

$$Y(n) \pm z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}$$

fractile of the std. normal distribution

- **For n small (n<30), use:**

$$Y(n) \pm t_{n-1,1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}$$

fractile of the t (student) distribution with n-1 d.f.

So, How Many Replications?

- **W = UB – LB is the width of the confidence interval, centered around Y(n)**
- **A measure of the relative estimation error is thus W / Y(n)**
- **So a good termination criteria is to set an estimation error β and run a number of replications n (or collect n data points) such that:**

$$W / Y(n) < \beta$$

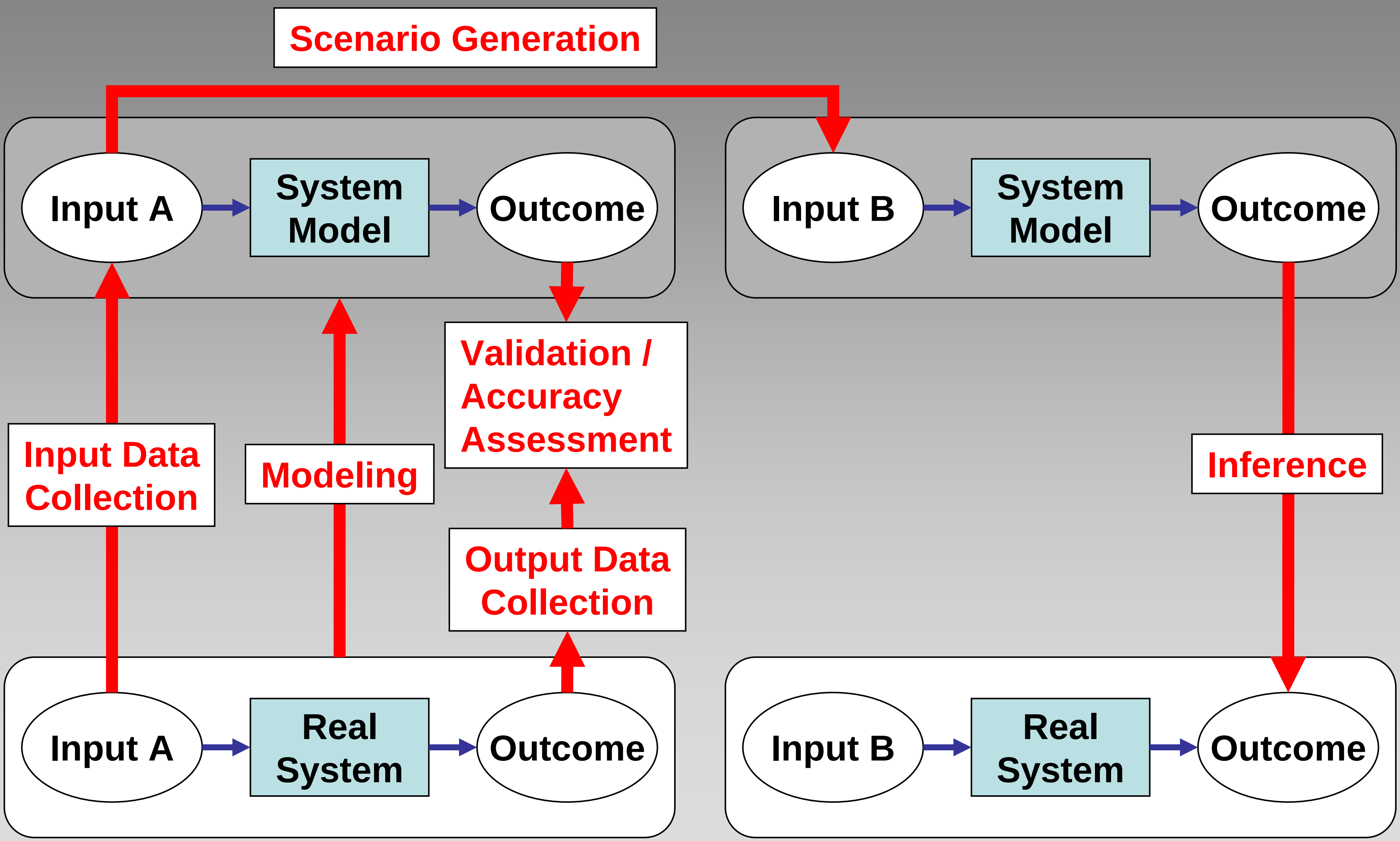
Example

Suppose we build a simulation model to assess the weekly cost of a proposed supply chain inventory policy

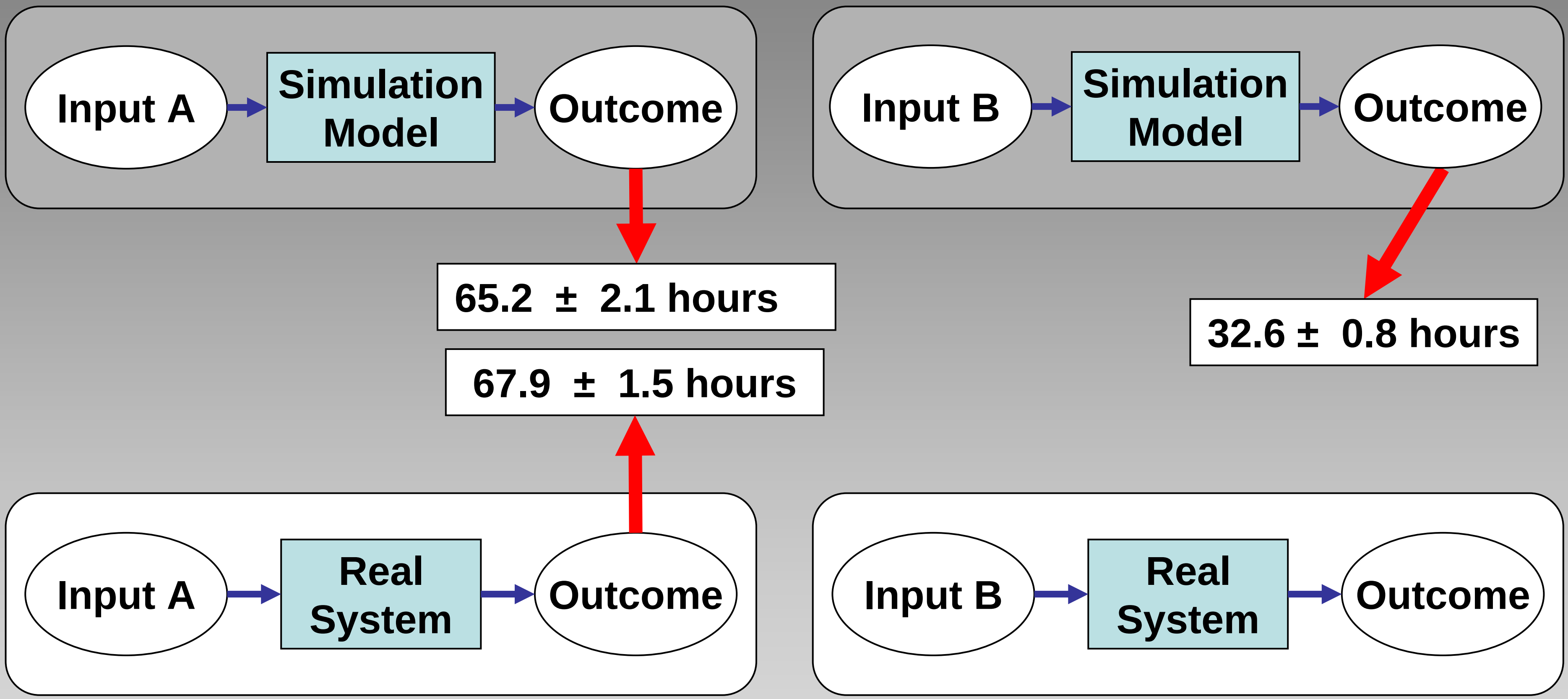
From the simulation data output (Spreadsheet “Confidence Interval”), we want to estimate:

1. **The average weekly cost C with Conf. Int.**
2. **P(C>$2M) with Conf. Int.**

The Simulation Process



Example: Cycle Time Simulation



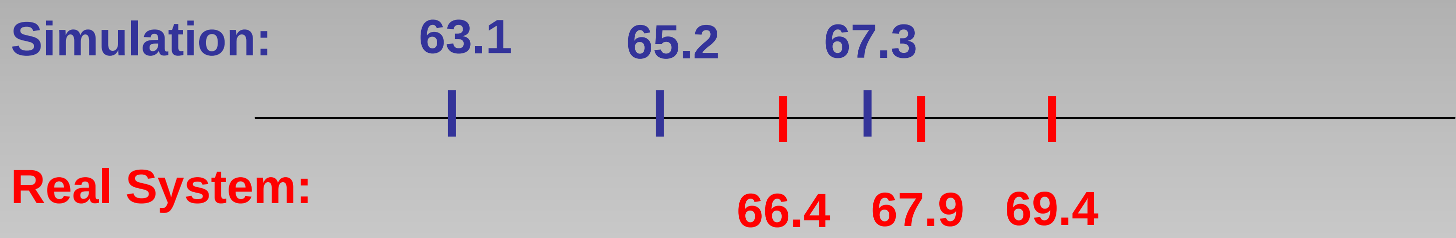
1. What is the estimation accuracy from the validation experiments?
2. What is the prediction accuracy of this simulation model?
3. What can you say about the likely result of Input B in the real system?

Cycle Time Simulation: Answers

1. What is the estimation accuracy from the validation experiments?

Simulation model: (2.1 x 2) / 65.2 = 6.4 %
Real system data: (1.5 x 2) / 67.9 = 4.4 %

2. What is the prediction accuracy of this simulation model?

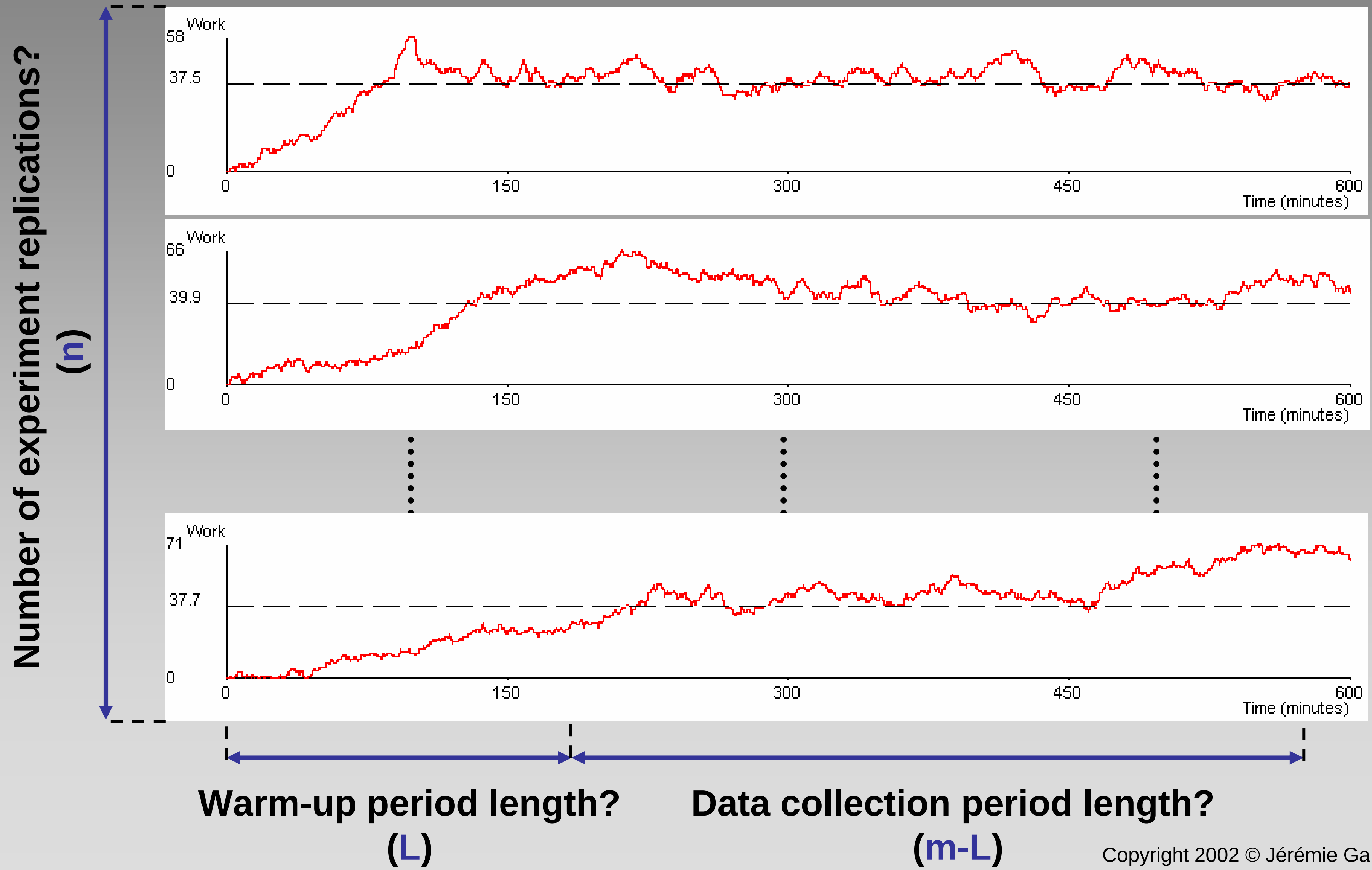


Prediction Accuracy: (69.4 – 63.1) / 65.2 = 9.6 %

3. What can you say about the likely result of Input B in the real system?

Should expect output between: (32.6 – 0.8)(1-0.096) = 28.7
and: (32.6 + 0.8)(1.096) = 36.6

Steady-State Analysis



Steady-State Analysis

- Let $Y_1, Y_2, Y_3, \ldots, Y_t, \ldots$ be a stochastic process (e.g. queue length at time index t).

  We want to estimate lim E[Y_t] when $t \to \infty$!

- Data Y_{ij} : j-th observation in the i-th replication

1[st] replication →
2[nd] replication →
⋮
last replication →

$$\begin{array}{l} Y_{11}, Y_{12}, Y_{13}, \cdots, Y_{1j}, \cdots, Y_{1m} \\ Y_{21}, Y_{22}, Y_{23}, \cdots, Y_{2j}, \cdots, Y_{2m} \\ \vdots \qquad\qquad\qquad\quad \vdots \qquad\; \vdots \\ Y_{n1}, Y_{n2}, Y_{n3}, \cdots, Y_{nj}, \cdots, Y_{nm} \end{array}$$

first observation ··················· last observation

Replication/Deletion Approach

(Y_{ij} : j-th observation in the i-th replication)

- For each replication i, instead of the estimator:

  $Y_i(m) = (Y_{i1}+Y_{i2}+Y_{i3}+ \ldots +Y_{im}) / m,$

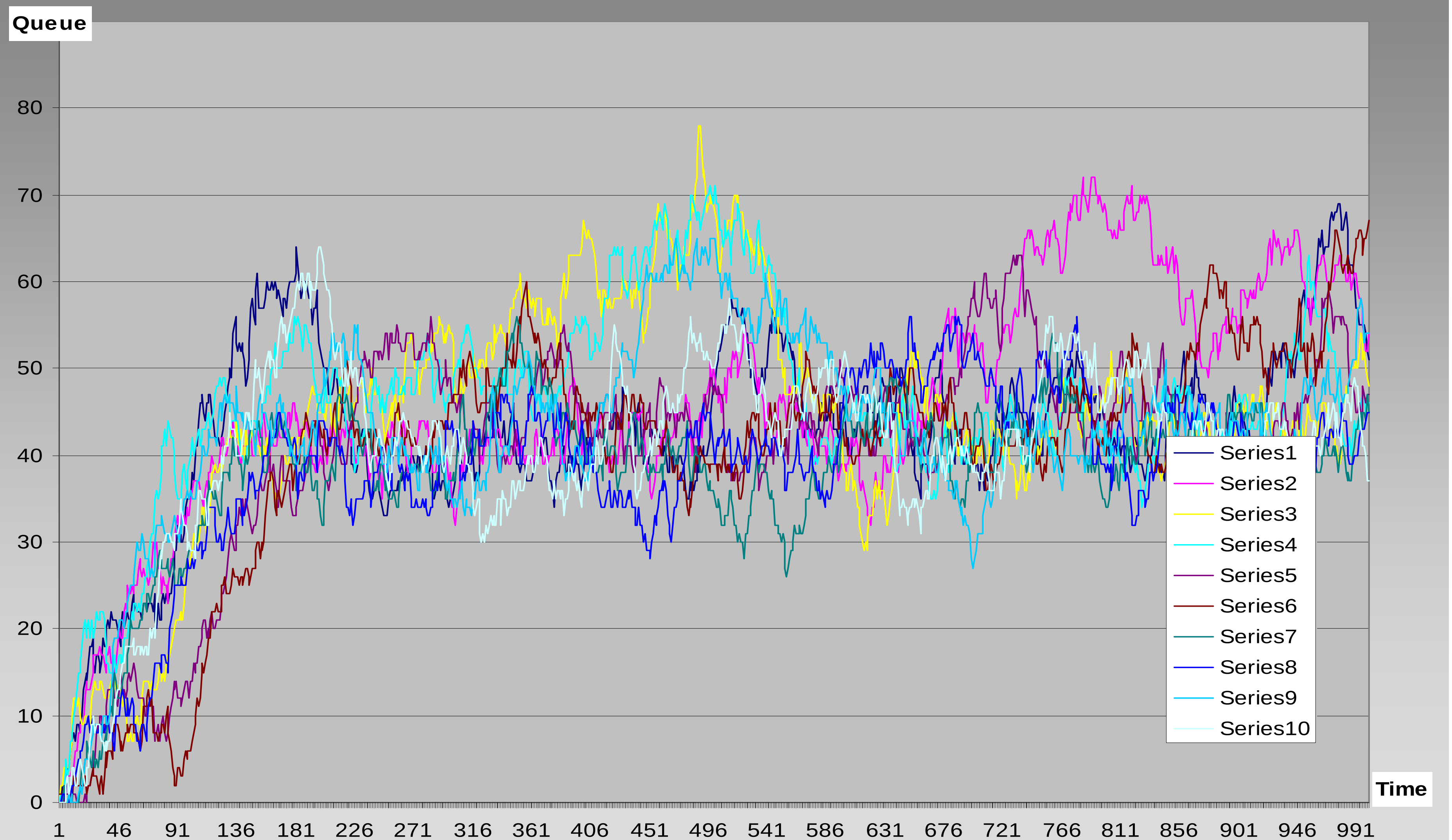
  consider the modified estimator:

$$Y_i(m,L) = (Y_{iL}+Y_{i(L+1)}+Y_{i(L+2)}+ \ldots +Y_{im}) / (m-L+1)$$

- An estimator for lim $E[Y_t]$ when $t \rightarrow \infty$ is then

$$Y(n) = (Y_1(m,L) + Y_2(m,L) \ldots + Y_n(m,L)) / n$$

Experimental Data Plot



Welch's Method for Warm-up

(Y_{ij} : j-th observation in the i-th replication)

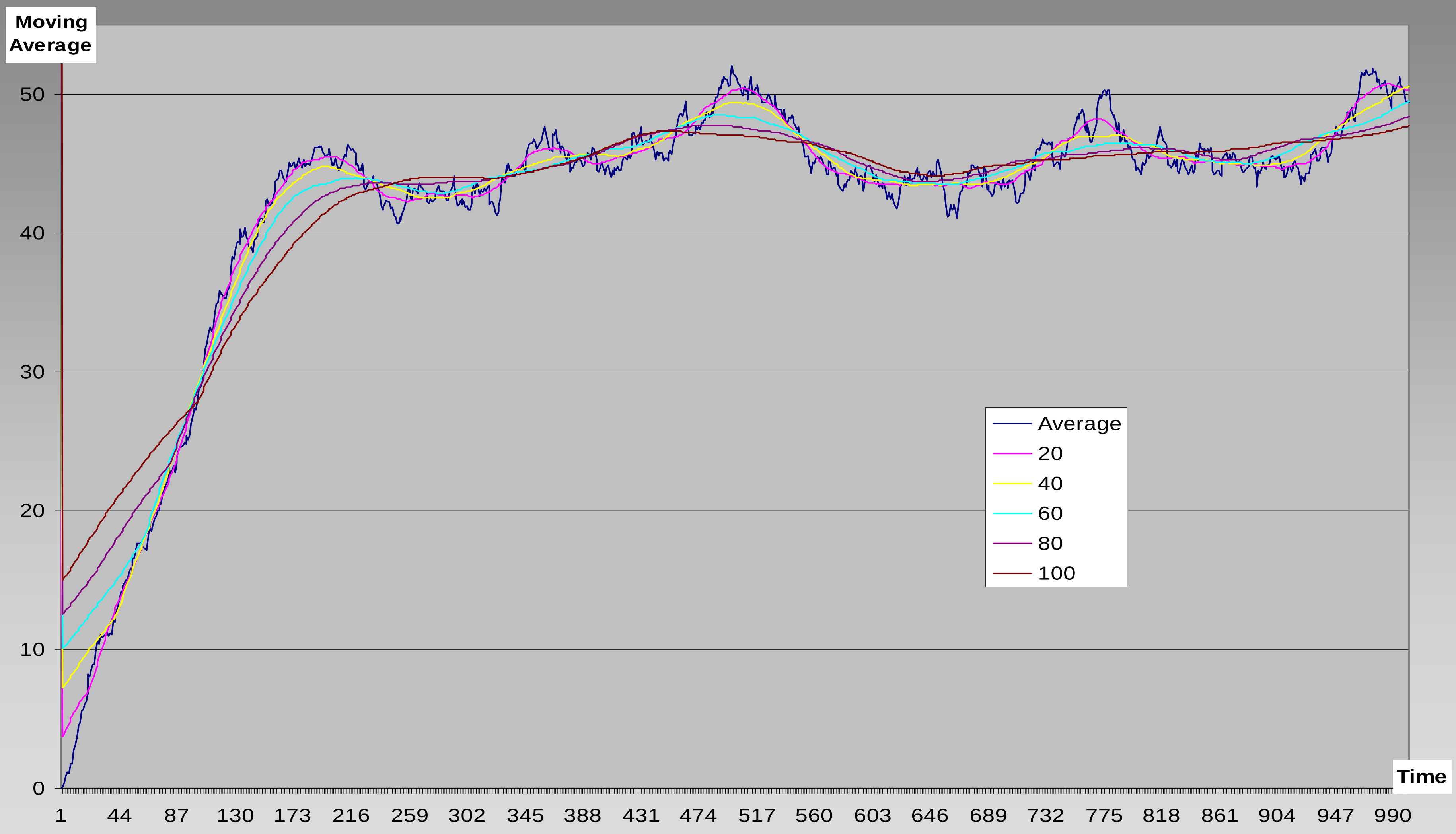
- **Compute the average across replications for each time point:**

$$Y_t[n] = (Y_{1t}+Y_{2t}+Y_{3t}+ \ldots +Y_{nt}\ ) / n$$

- **Welch's method is to plot the moving average process of $Y_t[n]$ based on various lags w:**

$$Y_t[n,w] = (Y_{t-w}[n] + \ldots + Y_t[n] + \ldots + Y_{t+w}[n]) / (2w + 1)$$

Welch's Method for Warm-up



Class 5 Wrap-Up

1. **Remember the Dice!**
2. **Construction of confidence intervals**
3. **The Wall**
4. **Use common sense for warm-up and data collection length**