Introduction to Modeling and Simulation: Test 1 Sample Questions

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1. The 2-D heat conduction equation is written:

$$\frac{\partial T}{\partial t} = \nu \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + f,$$

where ν is the thermal diffusivity ($=k/\rho c$) and f is temperature change due to heat generation ($=\dot{q}/\rho c$, $\dot{q}$ is the heat generation rate).

(a) Write this as a difference equation with explicit (*a.k.a.* forward Euler) timestepping on a Cartesian grid with uniform spacings Δx and Δy, and put all new temperatures (timestep $n+1$) on the left side and old temperatures (timesteps n) on the right.

(b) For the infinitely periodic grid of temperatures below, and assuming $\Delta x = \Delta y$ and $f = \dot{q} = 0$, compute the next two timesteps for mesh Fourier numbers ($\text{Fo}_M = \nu\Delta t/\Delta x^2$) of:

i. Fo_M=0.2
ii. Fo_M=0.3
iii. Fo_M=0.6

8°	10°	8°	10°
10°	8°	10°	8°
8°	10°	8°	10°
10°	8°	10°	8°

Initial temperatures at $t = t_0$, an infinite grid with repeating temperatures.

(c) What is the critical value of Fo_M for stability of this algorithm in two dimensions with $\Delta x = \Delta y$? (Note: it can be oscillating and not unstable, if the oscillations are shrinking instead of growing.)

(d) Is a 2-D explicit finite difference simulation always unstable for Fo_M just above your critical value? If so, explain why; if not, give an explanation or counterexample.

Solution

(a) For $T_{i,j,n} = T|_{x=x_i,y=y_j,t=t_n}$, replacing the PDE with the corresponding difference equation gives:

$$\frac{T_{i,j,n+1} - T_{i,j,n}}{\Delta t} = \nu \frac{T_{i-1,j,n} - 2T_{i,j,n} + T_{i+1,j,n}}{(\Delta x)^2} + \nu \frac{T_{i,j-1,n} - 2T_{i,j,n} + T_{i,j+1,n}}{(\Delta y)^2} + f_{i,j,n}.$$

It's easy to isolate $T_{i,j,n+1}$:

$$T_{i,j,n+1} = T_{i,j,n} + \frac{\nu\Delta t}{(\Delta x)^2}(T_{i-1,j,n} - 2T_{i,j,n} + T_{i+1,j,n}) + \frac{\nu\Delta t}{(\Delta y)^2}(T_{i,j-1,n} - 2T_{i,j,n} + T_{i,j+1,n}) + f_{i,j,n}\Delta t.$$

(b) With $\Delta x = \Delta y$ and $f = 0$, and substituting Fo_M for $\nu\Delta t/(\Delta x)^2$, this simplifies to:

$$T_{i,j,n+1} = T_{i,j,n} + \text{Fo}_M(T_{i-1,j,n} + T_{i+1,j,n} + T_{i,j-1,n} + T_{i,j+1,n} - 4T_{i,j,n}).$$

i. For $\text{Fo}_M = 0.2$, the cells with $T = 8°$ at t_0 have all four neighbors at 10°, so they become at t_1:

$$T_{n+1} = 8° + 0.2(4 \cdot 10° - 4 \cdot 8°) = 9.6°.$$

Likewise, the cells with $T = 10°$ at $t - 0$ become:

$$T_{n+1} = 10° + 0.2(4 \cdot 8° - 4 \cdot 10°) = 8.4°.$$

We can repeat these calculations moving forward, to get the sequence:

$$t_0 : 8°, 10°;\ t_1 : 9.6°, 8.4°;\ t_2 : 8.66, 9.36°;\ ...$$

This is oscillating but heading toward 9° everywhere, so it is **stable**.

ii. For $\text{Fo}_M = 0.3$, this becomes:

$$t_0 : 8°, 10°;\ t_1 : 10.4°, 7.6°;\ t_2 : 7.04, 10.96°;\ ...$$

This is clearly **unstable**.

iii. For $\text{Fo}_M = 0.6$, this is even worse:

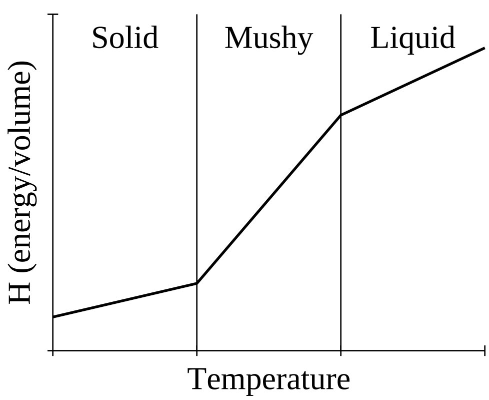
$$t_0 : 8°, 10°;\ t_1 : 12.8°, 5.2°;\ t_2 : -5.44, 23.44°;\ ...$$

(c) We get stable oscillations with the 8° nodes becoming 10° and vice versa when $\text{Fo}_M = 0.25$, so that is the stability criterion.

(d) The above temperature distribution is extreme. If the oscillations happen in 1-D, *i.e.* same as above but with each row the same as above and below it, then it is equivalent to 1-D and $\text{Fo}_M \leq 0.5$ is the stability criterion.

2. Enthalpy method and stability

When we add salt to water, there is no longer a single melting point but instead a "freezing range" over which the solution goes from fully solid to semi-solid ("mushy") to fully liquid. The enthalpy-temperature relationship looks something like:



Schematic relationship between enthalpy and temperature for salt water.

Given just this graph, which of the three regions (solid, mushy, liquid) is most likely to be stable, or unstable?

Solution:

The slope of the H vs. T curve is the product ρc, the volumetric heat capacity. Since the thermal diffusivity ν is the ratio $k/\rho c$ (where k is thermal conductivity), large ρc means small ν and more likely stable. Therefore, the mushy region is most likely stable, and the solid region least likely, and if the conductivities are roughly the same, one must choose the timestep satisfying the stability criterion in the solid in order to be stable everywhere.