Krane 5.1 Solution

Let's take extreme limit of the shell model: assume only the single unpaired nucleon determines the properties of the nucleus.

(a) ^{7}Li: A=7, Z=3, N=4
There is an unpaired proton at $1p_{3/2}$. Spin: 3/2, Parity: $(-1)^1=-1$.
Thus $I^{\pi}=3/2^-$

(b) ^{11}B: A=11, Z=5, N=6
There is an unpaired proton at $1p_{3/2}$. Spin: 3/2, Parity: $(-1)^1=-1$.
Thus $I^{\pi}=3/2^-$

(c) ^{15}C: A=15, Z=6, N=9
There is an unpaired neutron at $1d_{5/2}$. Spin: 5/2, Parity: $(-1)^2=1$.
Thus $I^{\pi}=5/2^+$

(d) ^{17}F: A=17, Z=9, N=8
There is an unpaired proton at $1d_{5/2}$. Spin: 5/2, Parity: $(-1)^2=1$.
Thus $I^{\pi}=5/2^+$

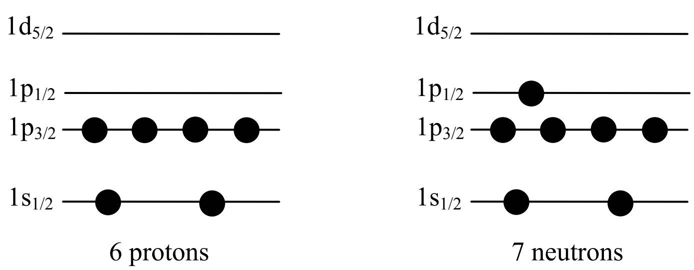
(e) ^{31}P: A=31, Z=15, N=16
There is an unpaired proton at $2s_{1/2}$. Spin: 1/2, Parity: $(-1)^0=1$.
Thus $I^{\pi}=1/2^+$

(f) ^{141}Pr: A=141, Z=59, N=82
There is an unpaired proton at $2d_{5/2}$. Spin: 5/2, Parity: $(-1)^2=1$.
Thus $I^{\pi}=5/2^+$

Krane 5.2 Solution

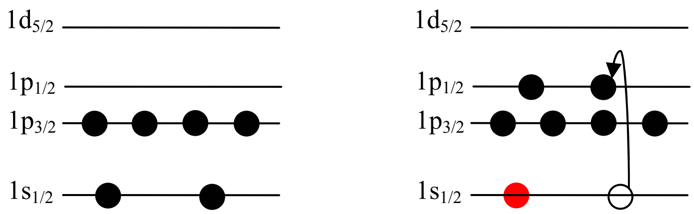
^{13}C: A=13, Z=6, N=7

In the **ground** state, all nucleons stay at the lowest possible energy level as follows:



The unpaired single neutron stays at $1p_{1/2}$, which gives $I^{\pi}=1/2^-$

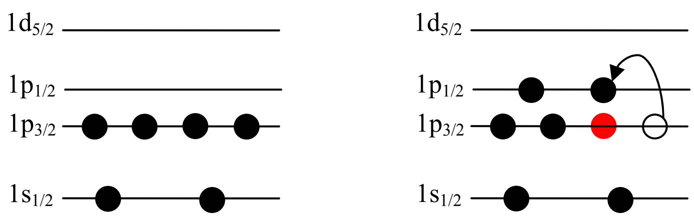
First excited state: $I^{\pi}=1/2^+$



6 protons 7 neutrons

The unpaired single neutron stays at $1s_{1/2}$, which gives $I^{\pi}=1/2^{+}$

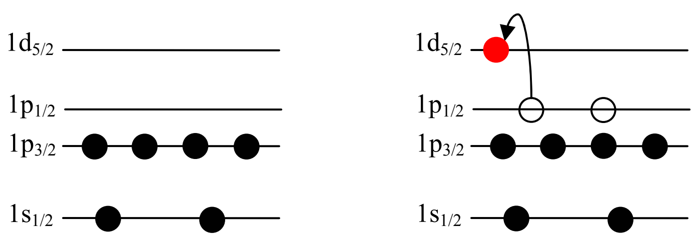
Second excited state: $I^{\pi}=3/2^{+}$



6 protons 7 neutrons

The unpaired single neutron stays at $1p_{3/2}$, which gives $I^{\pi}=3/2^{-}$

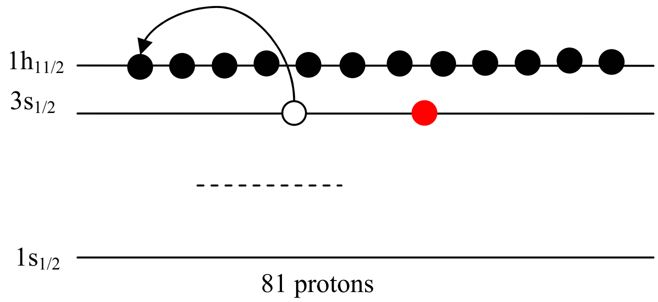
Third excited state: $I^{\pi}=5/2^{+}$



6 protons 7 neutrons

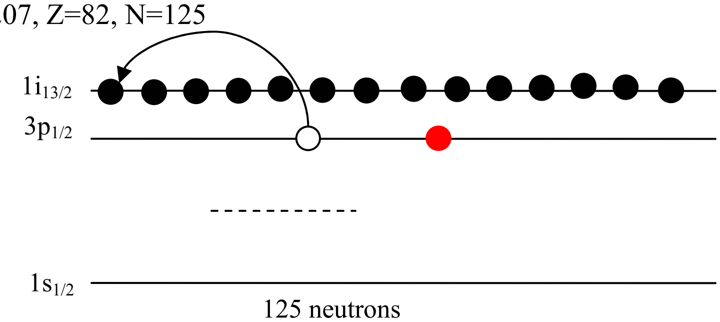
The unpaired single neutron stays at $1d_{5/2}$, which gives $I^{\pi}=5/2^{+}$

Krane 5.3 Solution
^{203}Tl: A=203, Z=81, N=122
We only consider the single unpaired nucleon in this problem. Check figure 5.6, we find the 81st proton should stay at $1h_{11/2}$, which means $I^{\pi}=11/2^{-}$ for ^{203}Tl. If the pairing force increases strongly with bigger l, it is more favorable in terms of lowering energy for a proton to stay at $1h_{11/2}$, making all the l=5 protons paired leaving a $3s_{1/2}$ proton unpaired, than it to stay at $3s_{1/2}$, leaving a $1h_{11/2}$ unpaired. This is shown in the following figure:



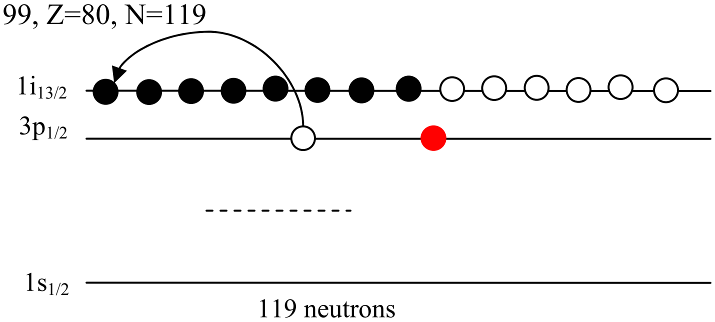
The unpaired protons is at $3s_{1/2}$, which gives $I^{\pi}=1/2^{+}$

^{207}Pb: A=207, Z=82, N=125



The unpaired protons is at $3p_{1/2}$, which gives $I^{\pi}=1/2^{-}$

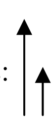
^{199}Hg: A=199, Z=80, N=119

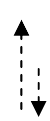


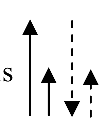
The unpaired protons is at $3p_{1/2}$, which gives $I^{\pi}=1/2^{-}$

Krane 5.5 Solution

^{16}N: A=16, Z=7, N=9, 2^{-}

Uncoupled neutron: $1d_{5/2}$, $l_n = 2$, $s_n = 1/2$ ➔ $j_n = 5/2 = l_n+s_n$, the diagram[1] is: 

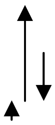
Uncoupled proton: $1p_{1/2}$, $l_p = 1$, $s_p = 1/2$ ➔ $j_p = ½ = l_p-s_p$, the diagram is: 

The total angular spin of the nucleus is $j = 2 = 5/2 - ½ = j_n-j_p$ and the diagram is 

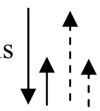
[1] The lengths of the arrows are not proportional to the numerical value of the angular momentum quantum number. Longer arrows always represent orbital angular momentum and shorter arrows are for spins. Also solid arrows indicate angular momentums for neutrons and dotted arrows for protons.

The parity of the nucleus is $(-1)^2(-1)^1 = -1$

^{12}B: A=12, Z=5, N=7, 1^+

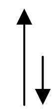
Uncoupled neutron: $1p_{1/2}$, $l_n = 1$, $s_n = 1/2$ ➔ $j_n = 1/2 = l_n - s_n$, the diagram is: 

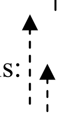
Uncoupled proton: $1p_{3/2}$, $l_p = 1$, $s_p = 3/2$ ➔ $j_p = 3/2 = l_p + s_p$, the diagram is: 

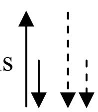
The total angular spin of the nucleus is $j = 1 = 3/2 – 1/2 = j_p - j_n$ and the diagram is 

The parity of the nucleus is $(-1)^1(-1)^1 = 1$

^{34}P: A=34, Z=15, N=19, 1^+

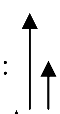
Uncoupled neutron: $1d_{3/2}$, $l_n = 2$, $s_n = 1/2$ ➔ $j_n = 3/2 = l_n - s_n$, the diagram is: 

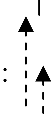
Uncoupled proton: $1s_{1/2}$, $l_p = 0$, $s_p = 1/2$ ➔ $j_p = 1/2 = l_p + s_p$, the diagram is: 

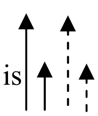
The total angular spin of the nucleus is $j = 1 = 3/2 – 1/2 = j_n - j_p$ and the diagram is 

The parity of the nucleus is $(-1)^2(-1)^0 = 1$

^{28}Al: A=28, Z=13, N=15, 3^+

Uncoupled neutron: $2s_{1/2}$, $l_n = 0$, $s_n = 1/2$ ➔ $j_n = 1/2 = l_n + s_n$, the diagram is: 

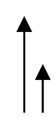
Uncoupled proton: $1d_{5/2}$, $l_p = 2$, $s_p = 1/2$ ➔ $j_p = 5/2 = l_p + s_p$, the diagram is: 

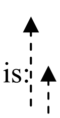
The total angular spin of the nucleus is $j = 3 = 5/2 + 1/2 = j_n + j_p$ and the diagram is 

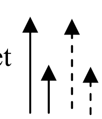
The parity of the nucleus is $(-1)^0(-1)^2 = 1$

From the above four diagrams, we observe that spin of the last unpaired neutron always aligns with spin of the last unpaired proton in a odd-odd nucleus. We apply this empirical rule to predict spin and parity of ^{26}Na and ^{28}Na as follows.

^{26}Na: A=26, Z=11, N=15

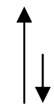
Uncoupled neutron: $2s_{1/2}$, $l_n = 0$, $s_n = 1/2$ ➔ $j_n = 1/2 = l_n + s_n$, the diagram is: 

Uncoupled proton: $1d_{2/5}$, $l_p = 2$, $s_p = 1/2$ ➔ $j_p = 5/2 = l_p + s_p$, the diagram is: 

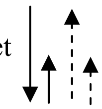
If we align spins of the neutron and proton, we get 

This means $j = j_n + j_p = 1/2 + 5/2 = 3$

The parity of the nucleus is $(-1)^0(-1)^2 = 1$

Thus, for ^{26}Na, $I^{\pi} = 3^{+}$
^{28}Na: A=26, Z=11, N=17
Uncoupled neutron: $1d_{3/2}$, $l_n = 2$, $s_n = 1/2$ ➔ $j_n = 3/2 = l_n - s_n$, the diagram is: 

Uncoupled proton: $1d_{5/2}$, $l_p = 2$, $s_p = 1/2$ ➔ $j_p = 5/2 = l_p + s_p$, the diagram is: 

If we align spins of the neutron and proton, we get 

This means $j = j_p - j_n = 5/2 - 3/2 = 1$

The parity of the nucleus is $(-1)^0(-1)^2 = 1$
Thus, for ^{28}Na, $I^{\pi} = 1^{+}$

Krane 5.6 Solution
(a) If we take into account the spin-orbit interaction, the nuclear energy level E_0 becomes:

$$V_{NUC} = E_0 + V_{so}\frac{\hat{l}\cdot\hat{s}}{\hbar^2} \qquad *$$

$\hat{l}\cdot\hat{s}$ is calculated from $\hat{j} = \hat{l} + \hat{s}$

$$\hat{l}\cdot\hat{s} = \frac{1}{2}[\hat{j}^2 - \hat{l}^2 - \hat{s}^2],\ j = l \pm s,\ s = \pm\frac{1}{2},\ \hat{j}^2 = j(j+1)\hbar^2, \hat{l}^2 = l(l+1)\hbar^2, \hat{s}^2 = s(s+1)\hbar^2$$

$$\hat{l}\cdot\hat{s} = \begin{array}{ll} \frac{\hbar^2}{2}\left[(l+\frac{3}{2})(l+\frac{1}{2}) - l(l+1) - \frac{3}{2}\times\frac{1}{2}\right] = \frac{l}{2}\hbar^2 & j = l + s = l + \frac{1}{2} \\ \frac{\hbar^2}{2}\left[(l+\frac{1}{2})(l-\frac{1}{2}) - l(l+1) - \frac{3}{2}\times\frac{1}{2}\right] = -\frac{(l+1)}{2}\hbar^2 & j = l - s = l - \frac{1}{2} \end{array}$$

From *, we get

$$V_{NUC1} = E_0 + \frac{l}{2}V_{so} \quad j_2 = l + \frac{1}{2}$$

$$V_{NUC2} = E_0 - \frac{(l+1)}{2}V_{so} \quad j_1 = l - \frac{1}{2}$$

(b) "Center of gravity" is the degeneracy weighted average energy of all the nucleons on the two split energy levels. It is calculated as follows:

$$\frac{V_{NUC1}D_1 + V_{NUC2}D_2}{D_1 + D_2} = \frac{(E_0 + \frac{l}{2}V_{so})\left[2(l+\frac{1}{2})+1\right] + (E_0 - \frac{(l+1)}{2}V_{so})\left[2(l-\frac{1}{2})+1\right]}{\left[2(l+\frac{1}{2})+1\right] + \left[2(l-\frac{1}{2})+1\right]} = E_0$$

Krane 5.14 Solution
For both ^{17}O and ^{19}O, 8 protons lead to a major shell closure, thus not contributing to the overall spin and parity of the nuclei. For ^{17}O, there are 9 neutrons with only one unpaired neutron with spin and parity of $1d_{5/2}$ staying outside the major shell closure. For ^{19}O, there are 11 neutrons

with 3 neutrons with spin and parity of $1d_{5/2}$ staying outside the major shell closure. If we only consider the last unpaired nucleon for both nuclei, they would have exactly the same overall spin and parity of $5/2^+$. In order to explain the difference, we should apply the many-particle shell model and look at different coupling possibilities of all the 3 neutrons at $1d_{5/2}$ for ^{19}O. For ^{19}O:

$\hat{\underline{j}} = \hat{\underline{j}}_1 + \hat{\underline{j}}_2 + \hat{\underline{j}}_3$, $m = m_1 + m_2 + m_3$

De to Pauli exclusive principle, $m_1 \neq m_2 \neq m_3$. Thus,

$$j_{\max} = m_{\max} = |m_1 + m_2 + m_3| = \frac{5}{2} + \frac{3}{2} + \frac{1}{2} = \frac{9}{2}$$

For $j = \frac{9}{2}$, $m = \pm\frac{9}{2}, \pm\frac{7}{2}, \pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$

$m = \frac{7}{2}$ can be only obtained by $\frac{5}{2} + \frac{3}{2} + \frac{1}{2}$, which is assigned to state $j = \frac{9}{2}$. This means there is no $j = \frac{7}{2}$ state.

$m = \frac{5}{2}$ can be obtained by $\frac{5}{2} + \frac{3}{2} - \frac{3}{2}$, which is assigned to state $j = \frac{9}{2}$
or by $\frac{5}{2} + \frac{1}{2} - \frac{1}{2}$, which is assigned to state $j = \frac{5}{2}$

$m = \frac{3}{2}$ can be obtained by $\frac{3}{2} + \frac{1}{2} - \frac{1}{2}$, which is assigned to state $j = \frac{9}{2}$
or by $\frac{5}{2} - \frac{5}{2} + \frac{3}{2}$, which is assigned to state $j = \frac{5}{2}$
or by $\frac{5}{2} + \frac{1}{2} - \frac{3}{2}$, which is assigned to state $j = \frac{3}{2}$

$m = \frac{1}{2}$ can be obtained by $\frac{5}{2} + \frac{3}{2} - \frac{1}{2}$, which is assigned to state $j = \frac{9}{2}$
or by $\frac{3}{2} + \frac{1}{2} - \frac{3}{2}$, which is assigned to state $j = \frac{5}{2}$
or by $\frac{5}{2} + \frac{1}{2} - \frac{5}{2}$, which is assigned to state $j = \frac{3}{2}$. This means there is no $j = \frac{1}{2}$ state.

In conclusion, $j = \frac{9}{2}, \frac{5}{2}, \frac{3}{2}$.

The parity of ^{19}O is $(-1)^2(-1)^2(-1)^2 = 1$.

From the above arguments, $\frac{3}{2}^+$ and $\frac{9}{2}^+$ can result from configuration of $(d_{5/2})^3$ for ^{19}O, which is not possible for ^{17}O.