1. Krane, problem 4.4. What fraction of the time do the neutron and proton in the deuteron spend beyond the range of their nuclear force?

Solution:

In center of mass spherical coordinate:

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla_r^{\ 2} + V_{NUC}(r)$$

The energy eigenvalue problem is written as:

$$[-\frac{\hbar^2}{2\mu}\nabla_r^{\ 2} + V_{NUC}(r)]\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi)$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi(r,\theta,\varphi)) + \frac{\hat{L}^2\psi(r,\theta,\varphi)}{2\mu r^2} + V_{NUC}(r)\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi)$$

Separate variables within ψ as $\psi(r,\theta,\varphi) = Y_l^m(\theta,\varphi)\phi(r)$ and notice $\hat{L}^2 Y_l^m = \hbar^2 l(l+1)Y_l^m$.

We get:

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{\hbar^2 l(l+1)}{2\mu r^2}\phi + V_{NUC}\phi = E\phi \qquad (1\text{-}1)$$

Let $u(r) = r\phi(r)$ (1-2) and $V_{eff} = V_{NUC} + \frac{\hbar^2 l(l+1)}{2\mu r^2}$

(1-1) becomes $-\frac{\hbar^2}{2\mu}\frac{\partial^2 u(r)}{\partial r^2} + (V_{eff} - E)u(r) = 0$

For ground state, l=0, $V_{eff}=V_{NUC}$

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2 u(r)}{\partial r^2} + (V_{NUC} - E)u(r) = 0 \qquad (1\text{-}3)$$

(1-3) has exactly the same form as one dimensional finite potential well problem. Solve (1-3), we get:

$r < R \qquad u(r) = A\sin k_1 r + B\cos k_1 r$

$r \geq R \qquad u(r) = Ce^{-k_2 r} + De^{k_2 r}$

Use boundary conditions,

at $r = 0$, $u(r) = r\phi(r) = 0 \Rightarrow B = 0$

at $r = +\infty$, $u(r) < +\infty \Rightarrow D = 0$

$$r < R \qquad u(r) = A\sin k_1 r \qquad k_1 = \sqrt{\frac{2\mu(E + V_{NUC})}{\hbar^2}} \qquad (1\text{-}4)$$

$$r \geq R \qquad u(r) = Ce^{-k_2 r} \qquad k_2 = \sqrt{\frac{-2\mu E}{\hbar^2}} \qquad (1\text{-}5)$$

From (1-2), (1-4) and (1-5)

$$r < R \qquad \phi(r) = \frac{A\sin k_1 r}{r} \qquad k_1 = \sqrt{\frac{2\mu(E + V_{NUC})}{\hbar^2}} \qquad (1\text{-}6)$$

$$r \geq R \qquad \phi(r) = \frac{Ce^{-k_2 r}}{r} \qquad k_2 = \sqrt{\frac{-2\mu E}{\hbar^2}} \qquad (1\text{-}7)$$

Again, use boundary condition at r=R as:

$$\phi(r=R^-)=\frac{A\sin k_1 R}{R}=\phi(r=R^+)=\frac{Ce^{-k_2R}}{R}$$
$$\Rightarrow \frac{A}{C}=\frac{e^{-k_2R}}{\sin k_1 R} \qquad (1\text{-}8)$$

Thus, we get the three dimensional wave function as:
$\psi(r,\theta,\varphi)=Y_l^m(\theta,\varphi)\phi(r)$, where $\phi(r)$ is given by (1-6) and (1-7).

The probability that the neutron and proton in the deuteron stay beyond the range of their nuclear force is:

$$P=\frac{\iiint_{r>R}|\psi(r,\theta,\varphi)|^2\,d\vec{r}^{\,3}}{\iiint_{r>R}|\psi(r,\theta,\varphi)|^2\,d\vec{r}^{\,3}+\iiint_{r>R}|\psi(r,\theta,\varphi)|^2\,d\vec{r}^{\,3}}$$

$$=\frac{\int_0^\pi \sin\theta d\theta\int_0^{2\pi}|Y_l^m(\theta,\varphi)|^2\,d\varphi\int_R^{+\infty}|\frac{Ce^{-k_2r}}{r}|^2\,r^2dr}{\int_0^\pi \sin\theta d\theta\int_0^{2\pi}|Y_l^m(\theta,\varphi)|^2\,d\varphi\int_R^{+\infty}|\frac{Ce^{-k_2r}}{r}|^2\,r^2dr+\int_0^\pi \sin\theta d\theta\int_0^{2\pi}|Y_l^m(\theta,\varphi)|^2\,d\varphi\int_0^{R}|\frac{A\sin k_1 r}{r}|^2\,r^2dr}$$

Note that $d\vec{r}^{\,3}=\sin\theta r^2 d\theta d\varphi dr$

$$P=\frac{|C|^2\int_R^{+\infty}e^{-2k_2r}dr}{|C|^2\int_R^{+\infty}e^{-2k_2r}dr+|A|^2\int_0^R\sin^2 k_1 r dr}$$

$$=\frac{\left.\frac{1}{2k_2}e^{-2k_2r}\right|_R^{+\infty}}{\left.\frac{1}{2k_2}e^{-2k_2r}\right|_R^{+\infty}+\frac{|C|^2}{|A|^2}\left.\left(\frac{r}{2}-\frac{1}{2}\frac{1}{2k_1}\sin 2k_1 r\right)\right|_0^R}$$

$$\mu c^2=940MeV,\hbar c=200Mev,E=-2.2MeV,V_{NUC}=35MeV,R=2.1F$$

$$k_1=\sqrt{\frac{2\mu(E+V_{NUC})}{\hbar^2}}=\sqrt{\frac{2\mu c^2(E+V_{NUC})}{\hbar^2c^2}}=\sqrt{\frac{940MeV\times(-2.2+35)MeV}{(200MeVF)^2}}=0.88\,{}^1\!/_F$$

$$k_2=\sqrt{\frac{-2\mu E}{\hbar^2}}=\sqrt{\frac{-2\mu c^2E}{\hbar^2c^2}}=\sqrt{\frac{940MeV\times 2.2MeV}{(200MeVF)^2}}=0.23\,{}^1\!/_F$$

Finally, we get $P=62.5\%$. For 62.5% of the time, the neutron and proton in the deuteron stay out side the range of their nuclear force.

2. Show that the deuteron has no excited states.

Solution:

1) The excited state from higher order radial eigenfunction does not exist.

If we assume that the radial eigenfunction one order higher exists, there is at least ¾ of a wave length within the potential well. That is: $\frac{3}{4}\lambda < R$ (2-1)

From (2-1), we will get a minimum requirement for the depth of the potential well $V_{\min}$. If $|V_{NUC}| > V_{\min}$, we believe that there is a non-zero probability that an excited state due to higher order eigenfunction existing. Otherwise, there is no such excited state.

From (2-1), $\frac{3}{4}\lambda = \frac{3}{4}\times\frac{2\pi}{k_1} = \frac{3}{2}\pi\sqrt{\frac{\hbar^2}{2\mu V_{\min}}} = R$

$$\Rightarrow V_{\min} = (\frac{3}{2R})^2\pi^2\frac{\hbar^2}{2\mu} = \frac{9}{4\times 2.1^2}\pi^2\frac{200^2}{940} = 214MeV$$

$|V_{NUC}| = 35MeV < V_{\min}$, which means (2-1) cannot be satisfied.
Thus an excited state due to higher order eigenfunction does not exist.

2) The excited state from non-zero angular momentum does not exist.

Suppose $l = 1 \Rightarrow V_{eff} = V_{NUC} + \frac{\hbar^2 l(l+1)}{2\mu r^2} > V_{NUC} + \frac{\hbar^2 l(l+1)}{2\mu R^2}$

$$\Rightarrow |V_{eff}| < |V_{NUC} + \frac{\hbar^2\times 1\times(1+1)}{2\mu R^2}| = |-35MeV + \frac{(200MeVF)^2\times 1\times(1+1)}{940MeV\times 2.1^2 F^2}| = 17MeV$$

In order to hold at least ¼ wave length inside the well, we need at least a potential energy V, which satisfies $\frac{1}{4}\lambda < R$. This means:

$$\frac{1}{4}\frac{2\pi}{k_1} = \frac{1}{4}\frac{2\pi}{\sqrt{\frac{2\mu V}{\hbar^2}}} < R$$

$$\Rightarrow V > \frac{\pi^2\hbar^2}{8\mu R^2} = \frac{\pi^2\times(200MeVF)^2}{4\times 940MeV\times 2.1^2 F^2} \approx 25MeV$$

Thus $|V_{eff}| = 17MeV$ is not deep enough for excited state due to non-zero angular momentum .

3. Prove that the di-neutron cannot exist.
Solution:
Write the individual spin of the two neutrons as s_1 and s_2. $s_1=s_2=1/2$.
Due to Pauli exclusive principle for Fermions, these two spins cannot be parallel. This means the total spin s=0 with $\{m_1, m_2\}=\{1/2, -1/2\}$ or $\{m_1, m_2\}=\{-1/2, 1/2\}$, that is only singlet state could possibly exist. The potential energy for the singlet system is:

$$V_{NUC} = V_0 + \frac{V_1}{\hbar^2}\hat{s}_1\hat{s}_2$$

$$\hat{s}_1\hat{s}_2 = \frac{1}{2}[\hat{s}^2 - \hat{s}_1^{\,2} - \hat{s}_2^{\,2}] = \frac{1}{2}[\hbar^2\times 0\times 1 - \hbar^2\times\frac{1}{2}\times\frac{3}{2} - \hbar^2\times\frac{1}{2}\times\frac{3}{2}] = -\frac{3}{4}\hbar^2$$

If we assume it has the same V_0 and V_1 as n-p system,

$$V_{NUC} = V_0 + \frac{V_1}{\hbar^2}\hat{s}_1\hat{s}_2 = V_{\sin glet, n-p} = -25MeV$$

As shown in 2) of problem 2, we need at least 25MeV to bound the two-nucleon system, which is about the same as V_{NUC}. This tells us that the wave function cannot turn over at R_0 and the n-n system is not bound. In reality, if we use more precise number to go through the calculation, we would get a $|V_{NUC}|$ slightly less than V, the minimum amount of potential energy to bound the two-nucleon system. In another word, the singlet state is slightly unbound.

In conclusion, we can say that di-neutron cannot exist.