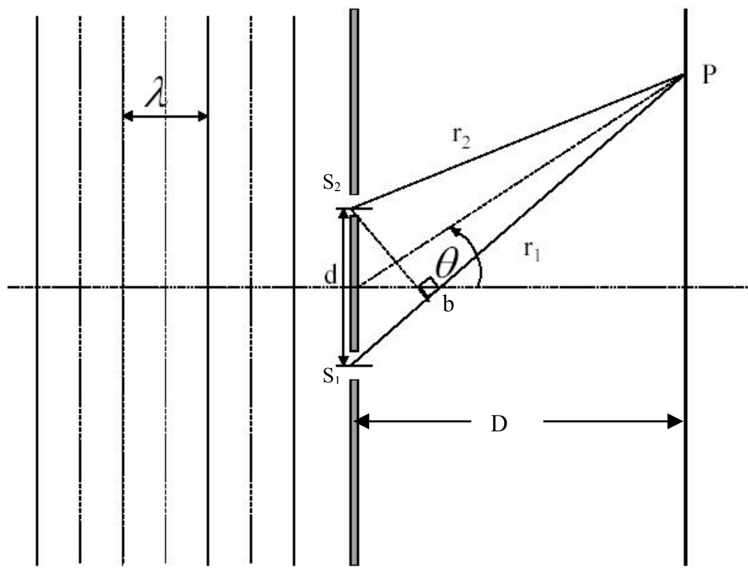
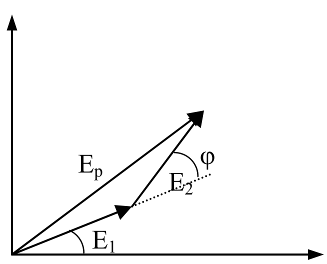
1. Wave Interference
Solution



Assume S_1 and S_2 are two identical sources. Their electric field components at point P can be expressed as:

$$E_1(t) = E\cos\omega t \qquad (1)$$

$$E_2(t) = E\cos(\omega t + \phi) \qquad (2)$$



Use phasor representation, the resulting interference wave electric field component is:

$$E_p{}^2 = E^2 + E^2 - 2E^2\cos(\pi - \phi)$$

$$E_p{}^2 = 2E^2 + 2E^2\cos\phi = 2E^2(1+\cos\phi)$$

$$E_p{}^2 = 4E^2\cos^2\frac{\phi}{2} \qquad (3)$$

The phase difference ϕ is associated with path difference as:

$$\frac{\phi}{2\pi} = \frac{r_2 - r_1}{\lambda}$$

If $D >> d$, $r_2 - r_1 \approx d\sin\theta \Rightarrow \phi \approx 2\pi\frac{d\sin\theta}{\lambda}$ (4)

Also we know the intensity of the wave is $I = \frac{E^2}{2 \mu_0 C}$

Thus $I_p = \frac{E_p^2}{2\mu_0 C}$

Insert (3), we get $I_p = \frac{E^2}{2\mu_0 C} \times 4\cos^2\frac{\phi}{2} = 4I_0\cos^2\frac{\phi}{2}$ (5)

In summary, $I_p = 4I_0\cos^2\frac{\phi}{2}$, $\phi \approx 2\pi\frac{d\sin\theta}{\lambda}$

2. Solution

(a) Transverse Wave: displacement of medium is perpendicular to the direction of travel of the wave. The displacement of a particle on the string at location x and time t can be expressed as: $y(x,t) = A\sin 2\pi(t/T + x/\lambda)$ (1)

Among them, A: amplitude, T: period, λ : wavelength, $k = \frac{2\pi}{\lambda}$: wave number,

$f = \frac{1}{T}$: frequency, $v = \frac{\lambda}{T}$: velocity.

Compare (1) with $y = 0.3\sin[\pi(0.5x - 50t)] = 0.3\sin[2\pi(0.25x - 25t)]$, we get

$A = 0.3cm$, $T = 0.04cm/s$, $\lambda = 4cm$, $k = \frac{\pi}{2}1/cm$, $f = 25(1/s)$,

$v = \frac{4cm}{0.04cm/s} = 100cm/s$

(b) The transverse speed of a particle at x is

$v(x,t) = \left|\frac{dy(x,t)}{dt}\right| = 15\pi\cos[\pi(0.5x - 50t)]$

The maximum speed is $15\pi cm/s$.

3. Liboff Problems

3.1 Solution

$$\hat{D}^2\varphi = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial^2\varphi}{\partial x^2} \Rightarrow \hat{D}^2 = \frac{\partial^2}{\partial x^2}$$

$$\hat{\Delta}^2\varphi = -\frac{\partial^2}{\partial x^2}\left(-\frac{\partial^2\varphi}{\partial x^2}\right) = \frac{\partial^4\varphi}{\partial x^4} \Rightarrow \hat{\Delta}^2 = \frac{\partial^4}{\partial x^4}$$

$$\hat{M}^2\varphi = \frac{\partial^2}{\partial x\partial y}\left(\frac{\partial^2\varphi}{\partial x\partial y}\right) = \frac{\partial^4\varphi}{\partial x^2\partial y^2} \Rightarrow \hat{M}^2 = \frac{\partial^4}{\partial x^2\partial y^2}$$ [1]

[1] For continuous f(x, y), $x \in X, y \in Y$, if both $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ exist, $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$.

$\hat{I}^2\varphi = \varphi \Rightarrow \hat{I}^2 =$ operation that leaves φ unchanged

$\hat{Q}^2\varphi = \int_0^1\int_0^1 \varphi dx' dx'' \Rightarrow \hat{Q}^2 = \int_0^1\int_0^1 dx' dx''$

$\hat{F}^2\varphi = F(x)F(x)\varphi = F^2(x)\varphi \Rightarrow \hat{F}^2 =$ multiplication by $F^2(x)$

$\hat{B}^2\varphi = \frac{1}{3}\left(\frac{1}{3}\varphi\right) = \frac{1}{9}\varphi \Rightarrow \hat{B}^2 =$ division by number 9

$\hat{\Theta}^2\varphi = \hat{\Theta}(0) = 0 \Rightarrow \hat{\Theta}^2 =$ operator that annihilates φ

$\hat{P}^2\varphi = \hat{P}(\varphi^3 - 3\varphi^2 - 4) = (\varphi^3 - 3\varphi^2 - 4)^3 - 3(\varphi^3 - 3\varphi^2 - 4)^2 - 4$

$\hat{G}^2\varphi = \hat{G}(8) = 8 \Rightarrow \hat{G}^2 =$ operator that changes φ to 8

3.2 Solution

$$\hat{D}^{-1}\hat{D}\varphi = \hat{D}^{-1}\left(\frac{\partial\varphi}{\partial x}\right) = \varphi$$

Though $\frac{\partial\varphi}{\partial x}$ could be inverted to φ by integration, the arbitrary integration constant makes it not unique. Thus general $\hat{D}^{-1}$ does not exist.

$\hat{I}^{-1}\hat{I}\varphi = \hat{I}^{-1}\varphi = \varphi$

$\Rightarrow \hat{I}^{-1} = \hat{I}$

$\hat{I}^{-1} =$ operation that leaves φ unchanged

$$\hat{F}^{-1}\hat{F}\varphi = \hat{F}^{-1}\left[F(x)\varphi\right] = \varphi = \frac{1}{F(x)}\left[F(x)\varphi\right],\ F(x) \neq 0$$

$$\Rightarrow \hat{F}^{-1} = \begin{cases} \dfrac{1}{F(x)} & F(x) \neq 0 \\ not \quad exists & F(x) = 0 \end{cases}$$

$\hat{B}^{-1}\hat{B}\varphi = \hat{B}^{-1} \times \frac{1}{3}\varphi = \varphi$

$\Rightarrow \hat{B}^{-1}\varphi = 3\varphi$

$\hat{\Theta}^{-1}\hat{\Theta}\varphi = \hat{\Theta}^{-1}(0) = \varphi$

$\Rightarrow$ There exists no $\hat{\Theta}^{-1}$

$\hat{G}^{-1}\hat{G}\varphi = \hat{G}^{-1}(8) = \varphi$

$\Rightarrow$ There exists no $\hat{G}^{-1}$

3.3 Solution

$$\hat{D}(a\varphi_1 + b\varphi_2) = \frac{\partial}{\partial x}(a\varphi_1 + b\varphi_2) = a\frac{\partial \varphi_1}{\partial x} + b\frac{\partial \varphi_2}{\partial x} = a\hat{D}\varphi_1 + b\hat{D}\varphi_1$$

$\Rightarrow \hat{D}$ is linear.

$$\hat{\Delta}(a\varphi_1 + b\varphi_2) = -\frac{\partial^2}{\partial x^2}(a\varphi_1 + b\varphi_2) = -a\frac{\partial^2 \varphi_1}{\partial x^2} - b\frac{\partial^2 \varphi_2}{\partial x^2} = a\hat{\Delta}\varphi_1 + b\hat{\Delta}\varphi_1$$

$\Rightarrow \hat{\Delta}$ is linear.

$$\hat{M}(a\varphi_1 + b\varphi_2) = \frac{\partial^2}{\partial x \partial y}(a\varphi_1 + b\varphi_2) = a\frac{\partial^2 \varphi_1}{\partial x \partial y} + b\frac{\partial^2 \varphi_2}{\partial x \partial y} = a\hat{M}\varphi_1 + b\hat{M}\varphi_1$$

$\Rightarrow \hat{M}$ is linear.

$$\hat{I}(a\varphi_1 + b\varphi_2) = a\varphi_1 + b\varphi_2 = a\hat{I}\varphi_1 + b\hat{I}\varphi_2$$

$\Rightarrow \hat{I}$ is linear.

$$\hat{Q}(a\varphi_1 + b\varphi_2) = \int_0^1 (a\varphi_1 + b\varphi_2)dx' = a\int_0^1 \varphi_1 dx' + b\int_0^1 \varphi_2 dx' = a\hat{Q}\varphi_1 + b\hat{Q}\varphi_2$$

$\Rightarrow \hat{Q}$ is linear.

$$\hat{F}(a\varphi_1 + b\varphi_2) = F(x)\left[a\varphi_1 + b\varphi_2\right] = aF(x)\varphi_1 + bF(x)\varphi_2 = a\hat{F}\varphi_1 + b\hat{F}\varphi_2$$

$\Rightarrow \hat{F}$ is linear.

$$\hat{B}(a\varphi_1 + b\varphi_2) = \frac{1}{3}\left[a\varphi_1 + b\varphi_2\right] = \frac{1}{3}a\varphi_1 + \frac{1}{3}b\varphi_2 = a\hat{B}\varphi_1 + b\hat{B}\varphi_2$$

$\Rightarrow \hat{B}$ is linear.

$$\hat{\Theta}(a\varphi_1 + b\varphi_2) = 0 = a\hat{\Theta}\varphi_1 + b\hat{\Theta}\varphi_2$$

$\Rightarrow \hat{\Theta}$ is linear.

$$\hat{P}(a\varphi_1 + b\varphi_2) = \left[a\varphi_1 + b\varphi_2\right]^3 - 3\left[a\varphi_1 + b\varphi_2\right]^2 - 4$$

$$a\hat{P}\varphi_1 + b\hat{P}\varphi_2 = a\varphi_1^3 - 3a\varphi_1^2 - 4a + b\varphi_2^3 - 3b\varphi_2^2 - 4b \neq \hat{P}(a\varphi_1 + b\varphi_2)$$

$\Rightarrow \hat{P}$ is not linear.

$$\hat{G}(a\varphi_1 + b\varphi_2) = 8$$

$$a\hat{G}\varphi_1 + b\hat{G}\varphi_2 = 8a + 8b \neq \hat{G}(a\varphi_1 + b\varphi_2)$$

$\Rightarrow \hat{G}$ is not linear.

3.10 Solution

$(\Delta x)^2$

$=<(x-<x>)^2>$

$=<x^2-2<x>x+<x>^2>$

$=<x^2>-2<x><x>+<x>^2$

$=<x^2>-<x>^2$

In classic physics, $x^2=(x)^2$, implying $\hat{x^2}=(\hat{x})^2=x^2$ in quantum mechanics.

$<x^2>=\int_{-\infty}^{+\infty}\psi^* x^2\psi dx$

$\psi(x,t)=A\exp\left[\frac{-(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right)\exp(-i\omega_0 t),\ A^2=\frac{1}{a\sqrt{2\pi}}$

Let $\eta=(x-x_0)/a$, $x=a(\eta+\eta_0)$, $\eta_0=x_0/a$

$<x^2>=Aa^3\int_{-\infty}^{+\infty}(\eta+\eta_0)^2 e^{-\frac{\eta^2}{2}}d\eta$

$<x^2>=Aa^3\left[\int_{-\infty}^{+\infty}\eta^2 e^{-\frac{\eta^2}{2}}d\eta+2\eta_0\int_{-\infty}^{+\infty}\eta e^{-\frac{\eta^2}{2}}d\eta+\eta_0^2\int_{-\infty}^{+\infty}e^{-\frac{\eta^2}{2}}d\eta\right]$

$\int_{-\infty}^{+\infty}\eta^2 e^{-\frac{\eta^2}{2}}d\eta=-\int_{-\infty}^{+\infty}\eta e^{-\frac{\eta^2}{2}}d\left(-\frac{\eta^2}{2}\right)=-\int_{-\infty}^{+\infty}\eta d\left(e^{-\frac{\eta^2}{2}}\right)=-\eta e^{-\frac{\eta^2}{2}}\Big|_{-\infty}^{+\infty}+\int_{-\infty}^{+\infty}e^{-\frac{\eta^2}{2}}d\eta=\sqrt{2\pi}$

$\int_{-\infty}^{+\infty}\eta e^{-\frac{\eta^2}{2}}d\eta=-\int_{-\infty}^{+\infty}e^{-\frac{\eta^2}{2}}d\left(-\frac{\eta^2}{2}\right)=-\int_{-\infty}^{+\infty}d\left(e^{-\frac{\eta^2}{2}}\right)=-e^{-\frac{\eta^2}{2}}\Big|_{-\infty}^{+\infty}=0$

$<x^2>=Aa^3\left[\sqrt{2\pi}+\eta_0^2\sqrt{2\pi}\right]=\frac{1}{a\sqrt{2\pi}}a^3\left[\sqrt{2\pi}+\eta_0^2\sqrt{2\pi}\right]=a^2+a^2\eta_0^2$

$<\Delta x>^2=<x^2>-<x>^2=a^2+a^2\eta_0^2-a^2\eta_0^2=a^2$

We can conclude that a is the spread, or, standard deviation of the underlying probability density distribution function of x. The bigger a is, the wider the shape of probability density distribution function $P(x)=|\psi|^2$ is.

3.12 Solution

(a) The possible outcomes are: 1, 2, 3, 4, 5, 6. Each of them has a probability of 1/6.

$$<s>=\sum_{k=1}^{6}P_k k=\frac{1}{6}[1+2+3+4+5+6]=\frac{7}{2}$$

(b) $\Delta s^2 = < s^2 > - < s >^2$

$$< s^2 > = \sum_{k=1}^{6} P_k k^2 = \frac{1}{6}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right] = \frac{91}{6}$$

$$\Delta s = \sqrt{< s^2 > - < s >^2} = \sqrt{\frac{91}{6} - \left(\frac{7}{2}\right)^2} = \frac{\sqrt{105}}{6}$$