

## 1. Particle in 1D box

a. The Eigenvalue equation within the box is:

$$\hat{H}\varphi_n = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi_n = E_n \varphi_n \quad (1.1)$$

Since  $\varphi_n(x) = 0$  outside the box and it is a continuous function, it should satisfy the following boundary conditions:

$$\varphi_n(0) = \varphi_n(L) = 0 \quad (1.2)$$

Rewrite (1.1) as:

$$\frac{\partial^2 \varphi_n}{\partial x^2} + k_n^2 \varphi_n = 0 \quad (1.3)$$

$$k_n^2 = \frac{2mE_n}{\hbar^2} \quad (1.4)$$

The solution to (1.3) is:

$$\varphi_n = A \sin k_n x + B \cos k_n x \quad (1.5)$$

Use boundary conditions in (1.2), we get:

$$\varphi_n(0) = B = 0 \Rightarrow \varphi_n(x) = A \sin k_n x, \text{ and} \quad (1.6)$$

$$\varphi_n(L) = A \sin k_n L = 0 \Rightarrow k_n L = n\pi, \quad n = 0, 1, 2, \dots \quad (1.7)$$

To get  $A$ , we normalize  $\varphi_n(x)$  as:

$$\begin{aligned} \int_0^L \varphi_n(x)^2 dx &= A^2 \int_0^L \sin^2 k_n x dx = A^2 \int_0^L \sin^2 \left( \frac{n\pi x}{L} \right) dx \\ \frac{A^2 L}{n\pi} \int_0^L \sin^2 \left( \frac{n\pi x}{L} \right) d \left( \frac{n\pi x}{L} \right) &= \frac{A^2 L}{n\pi} \int_0^{n\pi} \sin^2 \theta d\theta = \frac{A^2 L}{n\pi} \int_0^{n\pi} \left( \frac{1 - \cos 2\theta}{2} \right) d\theta = \frac{A^2 L}{2} = 1 \\ \Rightarrow A &= \sqrt{\frac{2}{L}} \end{aligned} \quad (1.8)$$

Put (1.5), (1.6), (1.7) and (1.8) together, we get the Eigenfunctions as:

$$\varphi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right), \quad 0 \leq x \leq L, \quad n = 0, 1, 2, \dots$$

$$\varphi_n(x) = 0, \quad x < 0 \text{ or } x > L$$

The energy spectrum  $E_n$ 's are:

$$E_n = \frac{k_n^2 \hbar^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \quad n = 0, 1, 2, \dots$$

b. The 1 dimension Eigenvalue equation for position is:

$$\hat{X}u(x) = xu(x) \quad (1.8)$$

The Eigenvalues are  $-\infty < x < +\infty$ , continuous.The Eigenfunctions are  $u(x) = \delta(x' - x)$ .The probability that a position measurement finds the particle within the interval  $\Delta x$  about  $x$ 

$$\text{is: } P(x)\Delta x = |c(x)|^2 \Delta x, \quad c(x) = \int_{-\infty}^{+\infty} u^*(x)\psi(x')dx' = \int_{-\infty}^{+\infty} \delta(x'-x)\psi(x')dx' = \psi(x)$$

Thus  $P(x)\Delta x = |\psi(x)|^2 \Delta x$

From a.,  $\psi(x) = \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ , which gives:

$$P(x)\Delta x = \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) \Delta x$$

c.

i) According to the first law of quantum mechanics, the possible measurement values of energy are:  $E_0, E_1, E_2, \dots$

According to the third law of quantum mechanics, the probability of finding  $E_n$  is

$P(E_n) = |c_n|^2$ ,  $c_n$  are the coefficients in the expansion of the wavefunction in terms of eigenfunctions of  $\hat{H}$ .

In this problem, the wavefunction is expanded as  $\psi(x) = a_1\psi_1(x) + a_3\psi_3(x)$ . From the

above laws, we know that the probability of getting  $E_1$  is  $P(E_1) = a_1^2$ , that of  $E_3$  is

$P(E_3) = a_3^2$  and there is no possibility of getting any other energy measurement value.

ii)  $a_1 = \pm\sqrt{P(E_1)} = \pm\sqrt{0.1} = \pm 0.3162$ ,  $a_3 = \sqrt{P(E_3)} = \pm\sqrt{0.9} = 0.9487$

$$\psi(x) = \pm 0.3162\psi_1(x) \pm 0.9487\psi_3(x) = \pm 0.3162\sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right) \pm 0.9487\sqrt{\frac{2}{L}} \sin\left(\frac{3\pi}{L}x\right)$$

iii) The probability of find the particle at  $x = \frac{2}{3}L$  is:

$$P\left(x = \frac{2}{3}L\right) = \left|\psi\left(x = \frac{2}{3}L\right)\right|^2$$

$$P\left(x = \frac{2}{3}L\right) = \left|\pm 0.3162\sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L} \frac{2}{3}L\right) \pm 0.9487\sqrt{\frac{2}{L}} \sin\left(\frac{3\pi}{L} \frac{2}{3}L\right)\right|^2$$

$$P\left(x = \frac{2}{3}L\right) = \left|\pm 0.3162\sqrt{\frac{2}{L}} \frac{\sqrt{3}}{2} \pm 0\right|^2 = 0.15/L$$

To be more precisely, the probability of finding the particle within  $\Delta x$  about  $\frac{2}{3}L$  is

$$P(x)\Delta x = \frac{0.15\Delta x}{L}$$

2. de Broglie wavelength

From  $p = \hbar k$  and  $\lambda = \frac{2\pi}{k}$ , we get  $\lambda = \frac{2\pi\hbar}{p}$ .

Also from  $E = \frac{p^2}{2m}$ , we know  $p = \sqrt{2mE}$ .

Combine the above two results, we get  $\lambda = \frac{2\pi\hbar}{\sqrt{2mE}}$ ,  $h = 2\pi\hbar = 4.1357 \times 10^{-15} \text{ eVs}$

or  $h = 2\pi\hbar = 6.6261 \times 10^{-34} \text{ Js}$

The de Broglie wavelength for a 10-eV electron is  $\lambda_e = \frac{4.1357 \times 10^{-15} \text{ eVs}}{\sqrt{2 \times (0.511 \text{ MeV} / c^2) \times 10 \text{ eV}}}$

$$\lambda_e = \frac{4.1357 \times 10^{-15} \text{ eVs} \times 3.0 \times 10^{10} \text{ cm/s}}{\sqrt{2 \times 0.511 \times 10^6 \text{ eV} \times 10 \text{ eV}}} = 3.88 \times 10^{-8} \text{ cm}$$

The de Broglie wavelength for a 10-MeV alpha particle is

$$\lambda_\alpha = \frac{4.1357 \times 10^{-15} \text{ eVs}}{\sqrt{2 \times 4 \times (931 \text{ MeV} / c^2) \times 10 \text{ MeV}}}$$

$$\lambda_\alpha = \frac{4.1357 \times 10^{-15} \text{ eVs} \times 3.0 \times 10^{10} \text{ cm/s}}{\sqrt{2 \times 4 \times 931 \times 10^6 \text{ eV} \times 10^7 \text{ eV}}} = 4.546 \times 10^{-13} \text{ cm}$$

The de Broglie wavelength for a 10g bullet, moving at a speed of 1,000m/sec is

$$\lambda_b = \frac{h}{p} = \frac{6.6261 \times 10^{-34} \text{ Js}}{10^{-2} \text{ Kg} \times 1000 \text{ m/s}} = 6.6261 \times 10^{-33} \text{ cm}$$

If the wavelength is  $10^{-8} \text{ cm}$ , the velocity of the bullet is:

$$v = \frac{p}{m} = \frac{h}{\lambda m} = \frac{6.6261 \times 10^{-34} \text{ Js}}{10^{-10} \text{ m} \times 10^{-2} \text{ Kg}} = 6.6 \times 10^{-22} \text{ m/s}$$

$$v = 6.6 \times 10^{-20} \text{ cm/s} \times \frac{\text{atomic\_displacements}}{10^{-8} \text{ cm}} \times 3600 \times 24 \times 365 \text{ s/year}$$

$$= 2.08 \times 10^{-4} \text{ atomic\_displacement/year}$$

### 3. Liboff, problem 7.42

Solution:

Suppose at  $x = x^*$  the particle density equals half of the particle density in the incident beam.

The wavefunction at region II is  $\varphi(x) = Ce^{-\kappa x}$

The particle density at  $x = 0$  is  $|\varphi(0)|^2 = |C|^2 e^{-\kappa \times 0} = |C|^2$

The particle density at  $x = x^*$  is  $|\varphi(x^*)|^2 = |C|^2 e^{-2\kappa x^*}$

$$\text{They should satisfy: } \frac{|\varphi(0)|^2}{|\varphi(x^*)|^2} = \frac{|C|^2}{|C|^2 e^{-2\kappa x^*}} = \frac{1}{e^{-2\kappa x^*}} = 2$$

$$\text{Thus we get: } x^* = \frac{\ln 2}{2\kappa} = \frac{\ln 2}{2\sqrt{2m(V-E)}} \hbar, V = 2E$$

$$\Rightarrow x^* = \frac{\ln 2\hbar}{2\sqrt{2mE}}$$

4. Tunneling probability

$$T \approx |e^{-\kappa L}|^2, \kappa = \sqrt{\frac{2m(V-E)}{\hbar^2}}$$

$$mc^2 \approx 1000\text{Mev}, \hbar^2 c^2 \approx (200\text{MevF})^2, V = 40\text{Mev}, E = 20\text{Mev}$$

$$\Rightarrow \kappa \approx \sqrt{\frac{2 \times 1000\text{Mev} \times (40 - 20)\text{Mev}}{(200\text{MevF})^2}} = 1\text{F}^{-1}$$

$$T \approx |e^{-1\text{F}^{-1} \times 10\text{F}}|^2 = e^{-20} = 2 \times 10^{-9}$$