

## 1. Solution

The energy eigenvalue problem for the given system is:

$$\hat{H}\psi = E\psi$$

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$V(x) = \begin{cases} 0 & |x| > a \\ -V_0 & |x| \leq a \end{cases}$$

For bounded particle, the solution is:

$$\psi_1 = Ae^{\kappa x} \quad x < -a$$

$$\psi_2 = B \sin kx \quad \text{or} \quad B' \cos kx \quad -a \leq x \leq a$$

$$\psi_3 = Ce^{-\kappa x} \quad x > a$$

$$k = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}, \quad \kappa = \sqrt{\frac{2mE}{\hbar^2}}$$

Since the solution has to be odd,  $\psi_2 = B \sin kx$

Also we know,  $\psi_2(a) = \psi_3(a)$ , if  $\psi_3(x)$  is decaying, there has to be at least a quarter of wave from 0 to  $a$  within the well. This means  $\lambda/4 \geq a$ .

“Just being bounded” means  $\lambda/4 = a$  and  $E = 0$ .

$$\text{This gives us } k = \frac{2\pi}{\lambda} = \frac{2\pi}{4a} = \sqrt{\frac{2mV_0}{\hbar^2}} \Rightarrow V_0 = \frac{\pi^2 \hbar^2}{8a^2 m}$$

## 2. Solution

Liboff 5.12

$$(a) \text{ Proof } [\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$$

$$LHS = [\hat{A} + \hat{B}, \hat{C}] = (\hat{A} + \hat{B})\hat{C} - \hat{C}(\hat{A} + \hat{B}) = \hat{A}\hat{C} + \hat{B}\hat{C} - \hat{C}\hat{A} - \hat{C}\hat{B}$$

$$RHS = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] = \hat{A}\hat{C} - \hat{C}\hat{A} + \hat{B}\hat{C} - \hat{C}\hat{B} = LHS$$

$$(b) \text{ Proof } [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

$$LHS = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B}$$

$$RHS = \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = LHS$$

Liboff 5.19

$$(\varphi_2, \varphi_3) = \{\sin kx, \exp(ikx)\}$$

Since we know  $\varphi_3 = \exp(ikx)$  is a common eigenfunction of  $\hat{H}$  and  $\hat{p}$ , we'd like to keep it as one of the two common eigenfunctions composed of  $(\varphi_2, \varphi_3)$ . If we denote it as  $\varphi_a$ ,

$$\text{then } \varphi_a = a_2\varphi_2 + a_3\varphi_3 = \varphi_3, \text{ which means } a_2 = 0 \text{ and } a_3 = 1. \quad (1)$$

Also we know  $\varphi_b = \exp(-ikx)$  is a common eigenfunction of  $\hat{H}$  and  $\hat{p}$  and is linearly independent of  $\varphi_a = \exp(ikx)$ , we hope we could construct  $\varphi_b$  as a linear combination of  $(\varphi_2, \varphi_3)$ .

$$\varphi_b = b_2\varphi_2 + b_3\varphi_3 = b_2 \sin kx + b_3 \exp(ikx) = b_2 \sin kx + b_3 (\cos kx + i \sin kx)$$

$$\begin{aligned}\varphi_b &= b_3 \cos kx + (b_2 + ib_3) \sin kx = \exp(-ikx) = \cos kx - i \sin kx \\ \Rightarrow b_3 &= 1, b_2 = -2i\end{aligned}\quad (2)$$

Sum (1) and (2), we get:

$$\varphi_a = 0 \times \varphi_2 + 1 \times \varphi_3 = \exp(ikx)$$

$$\varphi_b = -2i \times \varphi_2 + 1 \times \varphi_3 = \exp(-ikx)$$

$\varphi_2$  and  $\varphi_3$  are a pair of linearly independent common eigenfunctions of  $\hat{H}$  and  $\hat{p}$ .

### Liboff 5.25

Use uncertainty principle:

$$\text{If } [\hat{A}, \hat{B}] = C \neq 0, \text{ then } \Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$$

(a)  $\Delta x \Delta E$

$$[\hat{x}, \hat{E}] \psi = \hat{x} \hat{E} \psi - \hat{E} \hat{x} \psi$$

$$\hat{x} = x, \hat{E} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$[\hat{x}, \hat{E}] \psi = x \left[ -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi \right] - \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] [x \psi]$$

$$= -x \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + x V(x) \psi + \frac{\hbar^2}{2m} \frac{d^2}{dx^2} [x \psi] - x V(x) \psi$$

$$= -x \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{\hbar^2}{2m} \frac{d}{dx} \left[ \psi + x \frac{d\psi}{dx} \right] = -x \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{\hbar^2}{2m} \frac{d\psi}{dx} + x \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{\hbar^2}{2m} \frac{d\psi}{dx}$$

$$[\hat{x}, \hat{E}] = \frac{\hbar^2}{m} \frac{d}{dx}$$

$$\hat{p}_x = -i\hbar \frac{d}{dx} \Rightarrow [\hat{x}, \hat{E}] = i \frac{\hbar}{m} \hat{p}_x$$

$$\text{From uncertainty principle, we get } \Delta x \Delta E \geq \frac{1}{2} \left| \langle i \frac{\hbar}{m} \hat{p}_x \rangle \right|$$

$$\text{Thus } \Delta x \Delta E \geq \frac{1}{2} \frac{\hbar}{m} |\langle \hat{p}_x \rangle|$$

(b)  $\Delta p_x \Delta E$

$$\hat{p}_x = -i\hbar \frac{d}{dx}, \hat{E} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$[\hat{p}_x, \hat{E}] \psi = \hat{p}_x \hat{E} \psi - \hat{E} \hat{p}_x \psi$$

$$\begin{aligned}
[\hat{p}_x, \hat{E}]\psi &= -i\hbar \frac{d}{dx} \left[ -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi \right] - \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \left[ -i\hbar \frac{d\psi}{dx} \right] \\
&= i \frac{\hbar^3}{2m} \frac{d^3\psi}{dx^3} - i\hbar \frac{dV(x)}{dx} \psi - i\hbar \frac{d\psi}{dx} V(x) - i \frac{\hbar^2}{2m} \frac{d^3\psi}{dx^3} + i\hbar V(x) \frac{d\psi}{dx} \\
&= -i\hbar \frac{dV(x)}{dx} \psi \\
\Delta p_x \Delta E &\geq \frac{1}{2} \left| \langle -i\hbar \frac{dV(x)}{dx} \rangle \right| = \frac{\hbar}{2} \left| \langle \frac{dV(x)}{dx} \rangle \right|
\end{aligned}$$

(c)  $\Delta x \Delta T$ 

$$\begin{aligned}
\hat{x} &= x, \quad \hat{T} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \\
[\hat{x}, \hat{T}]\psi &= \hat{x}\hat{T}\psi - \hat{T}\hat{x}\psi \\
&= -\frac{\hbar^2 x}{2m} \frac{d^2\psi}{dx^2} + \frac{\hbar^2}{2m} \frac{d^2}{dx^2} (x\psi) = -\frac{\hbar^2 x}{2m} \frac{d^2\psi}{dx^2} + \frac{\hbar^2}{2m} \frac{d}{dx} (\psi + x \frac{d\psi}{dx}) \\
&= -\frac{\hbar^2 x}{2m} \frac{d^2\psi}{dx^2} + \frac{\hbar^2}{2m} \frac{d\psi}{dx} + \frac{\hbar^2}{2m} \frac{d\psi}{dx} + \frac{\hbar^2 x}{2m} \frac{d^2\psi}{dx^2} = \frac{\hbar^2}{m} \frac{d\psi}{dx} \\
[\hat{x}, \hat{T}] &= i \frac{\hbar}{m} \hat{p}_x \\
\Delta x \Delta T &\geq \frac{1}{2} \left| \langle i \frac{\hbar}{m} \hat{p}_x \rangle \right| = \frac{\hbar}{2m} \left| \langle \hat{p}_x \rangle \right|
\end{aligned}$$

(d)  $\Delta p_x \Delta T$ 

$$\begin{aligned}
\hat{p}_x &= -i\hbar \frac{d}{dx}, \quad \hat{T} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \\
[\hat{p}_x, \hat{T}]\psi &= \hat{p}_x \hat{T}\psi - \hat{T} \hat{p}_x \psi \\
&= -i\hbar \frac{d}{dx} \left( -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} \right) + \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \left( -i\hbar \frac{d\psi}{dx} \right) = i \frac{\hbar^3}{2m} \frac{d^3\psi}{dx^3} - i \frac{\hbar^3}{2m} \frac{d^3\psi}{dx^3} = 0 \\
\Delta p_x \Delta T &= 0
\end{aligned}$$

## 3. Solution

The 1D Schrodinger equation is:  $i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}\psi(x,t)$  (3.1)

Separate the variables of the solution as  $\psi(x,t) = \sum_n C_n(t) \varphi_n(x)$  (3.2)

Plug (3.2) into (3.1), we get:

$$i\hbar \sum_n \varphi_n(x) \frac{dC_n(t)}{dt} = \sum_n C_n(t) \hat{H} \varphi_n(x) \quad (3.3)$$

Insert  $\hat{H} \varphi_n(x) = E_n \varphi_n(x)$  into (3.3), we get:

$$i\hbar \sum_n \varphi_n(x) \frac{dC_n(t)}{dt} = \sum_n C_n(t) E_n \varphi_n(x)$$

$$\text{Match every term, } i\hbar \frac{dC_n(t)}{dt} = E_n C_n(t) \quad (3.4)$$

$$\text{Solve (3.4)} \Rightarrow C_n(t) = C_n(0) e^{-i\varpi_n t}, \quad \varpi_n = E_n / \hbar \quad (3.5)$$

$$\text{Replace } C_n(t) \text{ in (3.2) with (3.5)} \Rightarrow \psi(x,t) = \sum_n C_n(0) \varphi_n(x) e^{-i\varpi_n t}, \quad \varpi_n = E_n / \hbar \quad (3.6)$$

Use initial condition,  $\psi(x,0) = a_1 \varphi_1(x) + a_3 \varphi_3(x)$ , we get:

$$C_1(0) = a_1, \quad C_3(0) = a_3, \quad C_i(0) = 0 \text{ for } i \neq 1,3 \quad (3.7)$$

Insert (3.7) into (3.6):

$$\Rightarrow \psi(x,t) = a_1 e^{-i\varpi_1 t} \varphi_1(x) + a_3 e^{-i\varpi_3 t} \varphi_3(x) \quad (3.8)$$

From problem set2, we know  $\varphi_n(x) = \sqrt{\frac{2}{L}} \sin k_n x$ ,  $k_n = \frac{\pi n}{L}$ , put into (3.8)

$$\Rightarrow \psi(x,t) = a_1 \sqrt{\frac{2}{L}} e^{-i\varpi_1 t} \sin k_1 x + a_3 \sqrt{\frac{2}{L}} e^{-i\varpi_3 t} \sin k_3 x$$

$$k_1 = \frac{\pi}{L}, \quad \varpi_1 = \frac{E_1}{\hbar} = \frac{\hbar \pi^2}{2mL^2}, \quad k_3 = \frac{3\pi}{L}, \quad \varpi_3 = \frac{E_3}{\hbar} = \frac{9\hbar \pi^2}{2mL^2}$$

The probability of finding the particle between  $x = \frac{L}{2}$  and  $x = \frac{L}{2} + \Delta x$  for small  $\Delta x$  is:

$$P(x)|_{x=L/2} \Delta x = |\psi(x,t)|_{x=L/2}^2 \Delta x$$

$$P(x)|_{x=L/2} \Delta x = \left| a_1 \sqrt{\frac{2}{L}} e^{-i\varpi_1 t} \sin k_1 x + a_3 \sqrt{\frac{2}{L}} e^{-i\varpi_3 t} \sin k_3 x \right|_{x=L/2}^2 \Delta x$$

$$\sin k_1 x|_{x=L/2} = \sin\left(\frac{\pi}{L} \times \frac{L}{2}\right) = 1, \quad \sin k_3 x|_{x=L/2} = \sin\left(\frac{3\pi}{L} \times \frac{L}{2}\right) = -1$$

$$P(x = L/2) \Delta x = \frac{2\Delta x}{L} |a_1 e^{-i\varpi_1 t} - a_3 e^{-i\varpi_3 t}|^2$$

If we write  $a_1$  and  $a_3$  as  $a_1 = |a_1| e^{i\theta_1}$  and  $a_3 = |a_3| e^{i\theta_3}$

$$\begin{aligned} P(x = L/2) \Delta x &= \frac{2\Delta x}{L} \left| |a_1| e^{-i(\varpi_1 - \theta_1)t} - |a_3| e^{-i(\varpi_3 - \theta_3)t} \right|^2 \\ &= \frac{2\Delta x}{L} (|a_1|^2 + |a_3|^2 - 2|a_1||a_3| \cos[(\varpi_1 - \theta_1) - (\varpi_3 - \theta_3)]) \end{aligned}$$

Probabilities of energy measurement:

$$\text{a) } P(E_1) = |a_1 \exp(-i\omega_1 t)|^2 = |a_1|^2$$

$$\text{b) } P(E_2) = 0$$

$$\text{c) } P(E_3) = |a_3 \exp(-i\omega_3 t)|^2 = |a_3|^2$$

As can be seen from the result, the energy measurement probabilities don't change over time.

## 4. Solution

If  $[\hat{H}, \hat{p}_x] = 0$ ,  $p_x$  conserves.

If  $[\hat{H}, \hat{p}_y] = 0$ ,  $p_y$  conserves.

If  $[\hat{H}, \hat{p}_z] = 0$ ,  $p_z$  conserves.

$$\hat{H} = \hat{T} + \hat{V} \text{ and } [\hat{T}, \hat{p}_x] = [\hat{T}, \hat{p}_y] = [\hat{T}, \hat{p}_z] = 0$$

$$[\hat{H}, \hat{p}_x] = [(\hat{T} + \hat{V}), \hat{p}_x] = [\hat{T}, \hat{p}_x] + [\hat{V}, \hat{p}_x] = [\hat{V}, \hat{p}_x]$$

According to the same logic,

$$[\hat{H}, \hat{p}_y] = [(\hat{T} + \hat{V}), \hat{p}_y] = [\hat{T}, \hat{p}_y] + [\hat{V}, \hat{p}_y] = [\hat{V}, \hat{p}_y]$$

$$[\hat{H}, \hat{p}_z] = [(\hat{T} + \hat{V}), \hat{p}_z] = [\hat{T}, \hat{p}_z] + [\hat{V}, \hat{p}_z] = [\hat{V}, \hat{p}_z]$$

$$(a) V(x, y, z) = ax + by^2 + cz^3$$

$$[\hat{H}, \hat{p}_x]\psi = [\hat{V}, \hat{p}_x]\psi = (ax + by^2 + cz^3) \left( -i\hbar \frac{\partial \psi}{\partial x} \right) + i\hbar \left[ a\psi + (ax + by^2 + cz^3) \frac{\partial \psi}{\partial x} \right] = i\hbar a\psi$$

$$[\hat{H}, \hat{p}_x] = i\hbar a \neq 0.$$

The momentum does not conserve in x direction.

$$[\hat{H}, \hat{p}_y]\psi = [\hat{V}, \hat{p}_y]\psi = (ax + by^2 + cz^3) \left( -i\hbar \frac{\partial \psi}{\partial y} \right) + i\hbar \left[ 2by\psi + (ax + by^2 + cz^3) \frac{\partial \psi}{\partial y} \right] = i2\hbar by\psi$$

$$[\hat{H}, \hat{p}_y] = 2i\hbar by \neq 0.$$

The momentum does not conserve in y direction.

$$[\hat{H}, \hat{p}_z]\psi = [\hat{V}, \hat{p}_z]\psi = (ax + by^2 + cz^3) \left( -i\hbar \frac{\partial \psi}{\partial z} \right) + i\hbar \left[ 3cz^2\psi + (ax + by^2 + cz^3) \frac{\partial \psi}{\partial z} \right] = 3i\hbar cz^2\psi$$

$$[\hat{H}, \hat{p}_z] = 3i\hbar cz^2 \neq 0.$$

The momentum does not conserve in z direction.

$$(b) V(x, y, z) = a \arctan(x/L)$$

$$[\hat{H}, \hat{p}_x]\psi = [\hat{V}, \hat{p}_x]\psi = a \arctan(x/L) \left( -i\hbar \frac{\partial \psi}{\partial x} \right) + i\hbar \left[ a\psi \frac{1/L}{1+(x/L)^2} + a \arctan(x/L) \frac{\partial \psi}{\partial x} \right]$$

$$[\hat{H}, \hat{p}_x] = i\hbar a \frac{1/L}{1+(x/L)^2} \neq 0.$$

The momentum does not conserve in x direction.

$$[\hat{H}, \hat{p}_y]\psi = [\hat{V}, \hat{p}_y]\psi = a \arctan(x/L) \left( -i\hbar \frac{\partial \psi}{\partial y} \right) + i\hbar \left[ a \arctan(x/L) \frac{\partial \psi}{\partial y} \right] = 0$$

The momentum conserves in y direction.

$$[\hat{H}, \hat{p}_z] = [\hat{V}, \hat{p}_z] = a \arctan(x/L) \left( -i\hbar \frac{\partial_y}{\partial z} \right) + i\hbar \left[ a \arctan(x/L) \frac{\partial_y}{\partial z} \right] = 0$$

The momentum conserves in z direction.

$$(c) V(x, y, z) = b \exp(-y^2 / d^2)$$

$$[\hat{H}, \hat{p}_x] \psi = [\hat{V}, \hat{p}_x] \psi = b \exp(-y^2 / d^2) \left( -i\hbar \frac{\partial \psi}{\partial x} \right) + i\hbar \left[ b \exp(-y^2 / d^2) \frac{\partial \psi}{\partial x} \right] = 0$$

$$[\hat{H}, \hat{p}_x] = 0.$$

The momentum does conserve in x direction.

$$[\hat{H}, \hat{p}_y] \psi = [\hat{V}, \hat{p}_y] \psi = b \exp(-y^2 / d^2) \left( -i\hbar \frac{\partial \psi}{\partial y} \right) + i\hbar \left[ b \exp(-y^2 / d^2) \frac{\partial \psi}{\partial y} + b \exp(-y^2 / d^2) \left( -\frac{2y}{d^2} \right) \psi \right] \neq 0$$

The momentum does not conserve in y direction.

$$[\hat{H}, \hat{p}_z] \psi = [\hat{V}, \hat{p}_z] \psi = b \exp(-y^2 / d^2) \left( -i\hbar \frac{\partial \psi}{\partial z} \right) + i\hbar \left[ b \exp(-y^2 / d^2) \frac{\partial \psi}{\partial z} \right] = 0$$

The momentum conserves in z direction.

### 5. Solution

$$\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m} \nabla^2 + \hat{V}, \quad \hat{P} \psi(x, y, z) = \psi(-x, -y, -z)$$

If  $[\hat{H}, \hat{P}] = 0$ , energy and parity operators have common eigenfunctions.

First, we can proof that  $[\hat{T}, \hat{P}] = 0$ . Then we check if  $[\hat{V}, \hat{P}] = 0$ . If it is true,  $\hat{H}$  and  $\hat{P}$  have common eigenfunctions.

Prove  $[\hat{T}, \hat{P}] = 0$

$$\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} = \hat{T}_x + \hat{T}_y + \hat{T}_z$$

$$\hat{P} \psi(x, y, z) = \psi(-x, -y, -z) = \hat{P}_x \hat{P}_y \hat{P}_z \psi(x, y, z)$$

If we can show  $[\hat{T}_x, \hat{P}_x] = [\hat{T}_y, \hat{P}_y] = [\hat{T}_z, \hat{P}_z] = 0$ ,  $[\hat{T}, \hat{P}] = 0$  is proved.

Show  $[\hat{T}_x, \hat{P}_x] = 0$

Since  $\psi(x, y, z)$  is a linear combination of odd functions and even functions, if we can prove

that  $[\hat{T}_x, \hat{P}_x] \psi_1(x) = 0$ , where  $\psi_1(x) = \psi_1(-x)$  and  $[\hat{T}_x, \hat{P}_x] \psi_2(x) = 0$ , where

$\psi_2(x) = -\psi_2(-x)$ , it is true that  $[\hat{T}_x, \hat{P}_x] \psi(x, y, z) = 0$ .

$$[\hat{T}_x, \hat{P}_x] \psi_1(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1(-x)}{\partial x^2} + \frac{\hbar^2}{2m} \hat{P}_x \frac{\partial^2 \psi_1(x)}{\partial x^2} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1(x)}{\partial x^2} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi_1(x)}{\partial x^2} = 0$$

$$[\hat{T}_x, \hat{P}_x]\psi_2(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_2(-x)}{\partial x^2} + \frac{\hbar^2}{2m} \hat{P}_x \frac{\partial^2 \psi_2(x)}{\partial x^2} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_2(x)}{\partial x^2} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi_2(x)}{\partial x^2} = 0$$

According to the same logic,  $[\hat{T}_y, \hat{P}_y] = [\hat{T}_z, \hat{P}_z] = 0$

Thus  $[\hat{T}, \hat{P}] = 0$

(a)  $V(x, y, z) = e^2 / r$

$$[V(x, y, z), \hat{P}]\psi = (e^2 / r)\psi(-x, -y, -z) - \hat{P}[(e^2 / r)\psi(x, y, z)]$$

$$= (e^2 / r)\psi(-x, -y, -z) - (e^2 / r)\psi(-x, -y, -z) = 0$$

Thus  $[\hat{H}, \hat{P}] = 0$

The energy operator and the parity operator have common eigenfunctions.

(b)  $V(x, y, z) = ax + by^2 + cz^3$

$$[V(x, y, z), \hat{P}]\psi = (ax + by^2 + cz^3)\psi(-x, -y, -z) - \hat{P}[(ax + by^2 + cz^3)\psi(x, y, z)]$$

$$= (ax + by^2 + cz^3)\psi(-x, -y, -z) - (-ax + by^2 - cz^3)\psi(-x, -y, -z) \neq 0$$

Thus  $[\hat{H}, \hat{P}] \neq 0$

The energy operator and the parity operator do not have common eigenfunctions.

(c)  $V(x, y, z) = axy$

$$[V(x, y, z), \hat{P}]\psi = (axy)\psi(-x, -y, -z) - \hat{P}[(axy)\psi(x, y, z)]$$

$$= (axy)\psi(-x, -y, -z) - (axy)\psi(-x, -y, -z) = 0$$

Thus  $[\hat{H}, \hat{P}] = 0$

The energy operator and the parity operator have common eigenfunctions.

## 6. Solution

### Liboff9.3

The possible values of an angular momentum measurement are the square roots of eigenvalues of the total angular momentum operator  $\hat{L}^2$ . They are  $L = \sqrt{\hbar^2 l(l+1)}$ ,  $l = 0, 1, 2, \dots$ . In another word,  $\hat{L}^2 \varphi_l^m = \hbar^2 l(l+1) \varphi_l^m$ .

From the problem, we know  $L = \hbar\sqrt{56}$ , which gives us  $l = 7$ , since

$$L = \hbar\sqrt{56} = \sqrt{\hbar^2 \times 7 \times (7+1)}$$

The possible values of an angular momentum projection measurement on x axis are the eigenvalues of  $\hat{L}_x$ . They are  $L_x = 7\hbar, 6\hbar, \dots, -6\hbar, -7\hbar$ .

The angle  $\theta$  is given by  $\cos \theta = L_x / L \Rightarrow \theta = \arccos(L_x / L)$ .

Thus  $\theta_{\min} = \arccos(L_{x, \max} / L) = \arccos(7\hbar / \hbar\sqrt{56}) = 23^\circ$

### Libiff9.25(a)

The solution of eigenvalue problems for  $\hat{L}^2$  and  $\hat{L}_z$  are:

$$\hat{L}^2 Y_l^m = \hbar^2 l(l+1) Y_l^m, \quad l = 0, 1, 2, \dots$$

$$\hat{L}_z Y_l^m = \hbar m Y_l^m, \quad m = -l, -l+1, \dots, l$$

This means the possible values of measurements of  $L$  are  $\sqrt{\hbar^2 l(l+1)}$  and those of  $L_z$  are  $m\hbar$ .

The given state wave function is expanded with  $Y_l^m$  as:

$$\psi(\theta, \phi, 0) = \frac{3}{\sqrt{26}} Y_1^1 + \frac{4}{\sqrt{26}} Y_7^3 + \frac{1}{\sqrt{26}} Y_7^1$$

The possible values of  $L$  and  $L_z$  are:

$(L, L_z) = \{(1,1) \quad (7,3) \quad (7,1)\}$ . The possibilities of these measurements are:

$$P(L=1, L_z=1) = \left| \frac{3}{\sqrt{26}} \right|^2 = \frac{9}{26}$$

$$P(L=7, L_z=3) = \left| \frac{4}{\sqrt{26}} \right|^2 = \frac{8}{13}$$

$$P(L=7, L_z=1) = \left| \frac{1}{\sqrt{26}} \right|^2 = \frac{1}{26}$$

#### Libiff9.30

$$\hat{L}^2 |m, l\rangle = \hbar^2 l(l+1) |m, l\rangle, \quad l = 0, 1, 2, \dots$$

$$\hat{L}_z |m, l\rangle = \hbar m |m, l\rangle, \quad m = -l, -l+1, \dots, l$$

The above eigenvalue problem solutions are true for both particles, which means:

$$\begin{pmatrix} \hat{L}_1^2 \\ \hat{L}_{1z} \\ \hat{L}_2^2 \\ \hat{L}_{2z} \end{pmatrix} |m_1 l_1\rangle |m_2 l_2\rangle = \begin{pmatrix} \hbar^2 l_1(l_1+1) \\ \hbar m_1 \\ \hbar^2 l_2(l_2+1) \\ \hbar m_2 \end{pmatrix} |m_1 l_1\rangle |m_2 l_2\rangle$$

The eigenvalues of the set of operators  $(\hat{L}_1^2, \hat{L}_{1z}, \hat{L}_2^2, \hat{L}_{2z})$  corresponding to the product eigenstate  $|m_1 l_1\rangle |m_2 l_2\rangle$  are:  $\{m_1 \hbar, \hbar^2 l_1(l_1+1), m_2 \hbar, \hbar^2 l_2(l_2+1)\}$ .

#### Libiff9.34

$$l = 1, \quad m = -1, \quad l_1 = 1, \quad l_2 = 1,$$

$$l_1 = 1 \Rightarrow m_1 = -1, 0, 1$$

$$l_2 = 1 \Rightarrow m_2 = -1, 0, 1$$

$$\text{Due to coupling, } m = m_1 + m_2 \Rightarrow (m_1, m_2) = (-1, 0) \text{ or } (0, -1)$$

$$\text{Thus, } |l_1 m_1 l_2 m_2\rangle = C_{-1,0} |1, -1, 1, 0\rangle + C_{0,-1} |1, 0, 1, -1\rangle$$

$$\text{Find the coefficients as: } \hat{L}_- |1, -1, 1, 1\rangle = 0 = \sqrt{2}(C_{-1,0} + C_{0,-1}) |1, -1\rangle_1 |1, -1\rangle_2$$

$$\Rightarrow C_{-1,0} = -C_{0,-1}$$

$$\text{To normalize, } |C_{-1,0}|^2 + |C_{0,-1}|^2 = 1$$

$$\Rightarrow C_{-1,0} = -C_{0,-1} = \pm \frac{1}{\sqrt{2}}$$

## 22.02 Problem Set 3 Solution

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$$|l_1 m_1 l_2 m_2\rangle = \pm \frac{1}{\sqrt{2}} |1, -1, 1, 0\rangle \pm \frac{1}{\sqrt{2}} |1, 0, 1, -1\rangle$$

The possible values of  $L_{1z}$  are  $-\hbar$  with probability of  $\frac{1}{2}$  and  $0$  with probability of  $\frac{1}{2}$ .

### Libiff 9.35

$$l = 2, \quad m = -2, \quad l_1 = 1, \quad l_2 = 1,$$

$$l_1 = 1 \Rightarrow m_1 = -1, 0, 1$$

$$l_2 = 1 \Rightarrow m_2 = -1, 0, 1$$

$$\text{Due to coupling, } m = m_1 + m_2 \Rightarrow (m_1, m_2) = (-1, -1)$$

$$\text{Thus, } |l_1 m_1 l_2 m_2\rangle = |1, -1, 1, -1\rangle$$

There is only one set of possible quantum numbers. The joint probability of finding the two electrons with  $L_{1z} = L_{2z} = -\hbar$  is one.