

Equations

- Slice Selection:
$$\text{ideal } O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z) \delta(z - z_s) dz$$
- Better Approximation:
$$O_{2D}(x,y) = \int_{\Omega} O_{3D}(x,y,z) \text{TopHat} \left(\frac{z - z_s}{1/2 \Delta z} \right) dz$$
- Radon Transform:
$$P(\theta, z) = \int \int O(x,y) \delta(x \cos(\theta) + y \sin(\theta) + z)$$
- Fourier Transform:
$$\tilde{g}(k) = \int G(x) e^{-ikx} dx$$
- Central Slice Theorem:
$$k_z = k_x \cos(\theta) + k_y \sin(\theta)$$
$$z = x \cos(\theta) + y \sin(\theta)$$
- Back Projection:
$$O_r(x,y) = \frac{1}{\pi} \int_0^{\pi} P(\theta, z) d\theta$$
- Map:
$$\tilde{O}_r(k_x, k_y) = \tilde{P}(\theta, k_z)$$

where $k_z = k_x \cos(\theta) + k_y \sin(\theta)$

More Organized Proof of The Central Slice Theorem

The PSF associated with the simple Bach projection is:

$$PSF|_{BF} = \frac{1}{r}$$

$$\therefore O_r(x,y) = O(x,y) \otimes \frac{1}{\sqrt{x^2 + y^2}}$$

where $O_r(x,y) = B\{P(\alpha,z)\}$

$$\text{and } B = \frac{1}{\pi} \int_0^{\pi} P(\alpha, x \cos(\alpha) + y \sin(\alpha)) d\alpha$$

$$\therefore \underbrace{O_r(x,y)}_{\Updownarrow} = \underbrace{O(x,y)}_{\Updownarrow} \otimes \underbrace{\frac{1}{\sqrt{x^2 + y^2}}}_{\Updownarrow}$$

$$\tilde{O}_r(k_x, k_y) = \tilde{O}(k_x, k_y) \cdot \frac{1}{|k|}$$

so

$$\tilde{O}(k_x, k_y) = |k| \tilde{O}_r(k_x, k_y)$$

More Organized Proof of The Central Slice Theorem

1.
$$P(\alpha, z) = \iint O(x, y) \delta(x \cos(\alpha) + y \sin(\alpha) - z) dx dy$$

2. Equate the z-axis with a tilted reference frame

$$x' \parallel z, y' \perp z$$

$$\therefore x = x' \cos(\alpha) - y' \sin(\alpha)$$

$$y = x' \sin(\alpha) + y' \cos(\alpha)$$

and

$$x' = x \cos(\alpha) + y \sin(\alpha)$$

3. Substitute #2 into #1 and change integral to $dx' dy'$ (still over all space)

$$P(\alpha, z) = \iint O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) \delta(x' - z) dx' dy'$$

4. Integrate along x' and note that z is only a point along the x' axis.

$$P(\alpha, x) = \iint O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) dy'$$

5. Fourier Transform along x'

$$\tilde{p}(\alpha, k_x) = \iint O(x' \cos(\alpha) - y' \sin(\alpha), x' \sin(\alpha) + y' \cos(\alpha)) e^{-ix' k_x} dx' dy'$$

More Organized Proof of The Central Slice Theorem

6. Transform back to the (x,y) coordinate system

$$\tilde{p}(\alpha, k_x) = \iint O(x,y) e^{-i(x \cos(\alpha) + y \sin(\alpha)) k_x} dx dy$$

7. Define the tilted k -space coordinate system.

$$k_x = k_x \cos(\alpha) - k_y \sin(\alpha)$$

$$k_y = k_x \sin(\alpha) + k_y \cos(\alpha)$$

8. Rewrite #6 as

$$\tilde{p}(\alpha, k_x) = \iint O(x,y) e^{-i(k_x \cos(\alpha) - k_y \sin(\alpha))x} e^{-i(k_x \sin(\alpha) + k_y \cos(\alpha))y} dx dy \Big|_{k_y=0}$$

$$\tilde{p}(\alpha, k_z) = \iint O(x,y) e^{-ik_x x} e^{-ik_y y} dx dy \Big|_{k_y=0}$$

$$= F_{2D} \{O(x,y)\} \Big|_{k_y=0}$$

