

22.106 Neutron Interactions and Applications (Spring 2005)
Lecture 8 (3/3/05)

The Neutron Transport Equation - A Balance of Distributions

References --

- J. J. Duderstadt and W. R. Martin, *Transport Theory* (Wiley Interscience, 1979).
G. I. Bell and S. Glasstone, *Nuclear Reactor Theory* (Van Nostrand Reinhold, 1970).
K. M. Case, F. deHoffmann, G. Placzek, *Introduction to The Theory of Neutron Diffusion*, vol. 1, LASL, 1953.
R. K. Osborn and S. Yip, *The Foundations of Neutron Transport Theory* (Gordon and Breach, 1966).
-

Neutron Transport Equation -- An Introduction

The most fundamental equation in the study of neutron interactions and migration in any media is the neutron transport equation. This equation has as its solution the time-dependent distribution function for neutrons in configuration-velocity (phase) space. Knowledge of this distribution function is sufficient to solve almost all problems of interest in reactor theory. However, most of the time one does not need to know the distribution function itself, the integrals over some of the phase-space coordinates such as velocity direction, energy, or position are usually sufficient.

We consider a brief derivation of the transport equation to show that it is just a balance relation. One can give a more elaborate derivation than what we do here, including a quantum mechanical formulation, but we believe this is not necessary for the basic understanding of the physical content of the equation. Define the number density as

$$n(\underline{r}, \underline{v}, t) d^3r d^3v \equiv \text{expected no. of neutrons in } d^3r \text{ about } \underline{r} \text{ with} \\ \text{velocities in } d^3v \text{ about } \underline{v} \text{ at time } t$$

Instead of the vector variable \underline{v} , it is often more convenient to use the scalar variable E and a two-dimensional vector $\underline{\Omega}$, so that $\underline{v} = (v_x, v_y, v_z) \rightarrow (v, \theta, \varphi) \rightarrow (E, \theta, \varphi)$.

Correspondingly, $d^3v = v^2 dv d\Omega$, with $d\Omega = \sin\theta d\theta d\varphi$. Then we have

$$n(\underline{r}, \underline{v}, t) d^3r d^3v \equiv n(\underline{r}, E, \underline{\Omega}, t) d^3r dE d\Omega = \text{expected no. neutrons in } d^3r \text{ about } \underline{r} \\ \text{with energies in } dE \text{ about } E \text{ and going in directions in } d\Omega \\ \text{about } \underline{\Omega} \text{ at time } t.$$

Consider a subsystem of volume V and surface S . Suppose we want to calculate the change in the number of neutrons in V with energies in dE about E and direction in $d\Omega$ about $\underline{\Omega}$ during a time interval Δt . This is given by

$$\int_V [n(\underline{r}, E, \underline{\Omega}, t + \Delta t) - n(\underline{r}, E, \underline{\Omega}, t)] d^3 r dE d\Omega = \text{Gains} - \text{Losses} \quad (8.1)$$

There are two contributions to the gains.

(1) *Fission and external source*

$$\frac{\nu f(E) dE d\Omega}{4\pi} \iiint_{V, \underline{\Omega}', E'} \Sigma_f(E') \phi(\underline{r}, E', \underline{\Omega}', t) dE' d\Omega' d^3 r \Delta t + \int_V S(\underline{r}, E, \underline{\Omega}, t) d^3 r dE d\Omega \Delta t \quad (8.2)$$

where $f(E)$ is the fission spectrum, $\phi(\underline{r}, E, \underline{\Omega}, t) \equiv n(\underline{r}, E, \underline{\Omega}, t)v$ is the *neutron flux*, and S is the external source distribution.

(2) *Scattering*

$$\iiint_{V, E', \underline{\Omega}'} \Sigma_s(E') \phi(\underline{r}, E', \underline{\Omega}', t) d^3 r dE' d\Omega' \Delta t F(E' \underline{\Omega}' \rightarrow E \underline{\Omega}) dE d\Omega \quad (8.3)$$

where $F(E' \underline{\Omega}' \rightarrow E \underline{\Omega}) dE d\Omega =$ conditional probability that given a neutron scattered at $(E', \underline{\Omega}')$, its energy will be in dE about E and its direction will be in $d\Omega$ about $\underline{\Omega}$.

For losses there are also two terms, one for collisions and the other convective flow.

(3) *Collisions*

$$\int_V \Sigma_t(E) \phi(\underline{r}, E, \underline{\Omega}, t) d^3 r dE d\Omega \Delta t$$

(4) *Net Flow Outward*

$$\int_S \underline{\Omega} \cdot \hat{n} v n(\underline{r}_s, E, \underline{\Omega}, t) ds dE d\Omega \Delta t = \int_V d^3 r \underline{\Omega} \cdot \underline{\nabla} \phi(\underline{r}, E, \underline{\Omega}, t) dE d\Omega \Delta t \quad (8.4)$$

where \hat{n} is the outward normal at \underline{r}_s , and the divergence theorem, $\int_S d\underline{s} \cdot \underline{F} = \int_V d^3 r \underline{\nabla} \cdot \underline{F}$,

has been applied. Putting together the gains and losses, dividing by Δt , and taking the limit of $\Delta t \rightarrow 0$, we can write the balance, Eq. (8.1), as $\int_V [] = 0$. Since V can be any

arbitrary part of the system, the integrand $[]$ must vanish identically if the integral is to vanish for any V . Thus,

$$\frac{\partial n(\underline{r}, E, \underline{\Omega}, t)}{\partial t} = \frac{\nu f(E)}{4\pi} \int_{E', \underline{\Omega}'} dE' d\Omega' \Sigma_f(E') \phi(\underline{r}, E', \underline{\Omega}', t) + S(\underline{r}, E, \underline{\Omega}, t) \quad (8.5)$$

$$+ \int_{E, \underline{\Omega}} dE' d\Omega' \Sigma_s(E') \phi(\underline{r}, E', \underline{\Omega}', t) F(E' \underline{\Omega}' \rightarrow E \underline{\Omega}) - \Sigma_t(E) \phi(\underline{r}, E, \underline{\Omega}, t) - \underline{\Omega} \cdot \nabla \phi(\underline{r}, E, \underline{\Omega}, t)$$

Eq.(8.5) is what is known as the neutron transport equation for a homogeneous medium. For a heterogeneous system we simply let $\Sigma(E) \rightarrow \Sigma(\underline{r}, E)$. Notice that this is a *linear* equation because we have ignored the neutron-neutron interaction (the mean free path for such events is 10^8 cm or greater). Sometimes the neutron transport equation is also called the Boltzmann equation; one should be careful when using this terminology since the Boltzmann equation in kinetic theory of gases treats explicitly the collisions among the particles and is in general *nonlinear*.

The transport equation is an integro-differential equation in 7 variables. While it is much too complicated for us to attempt any kind of solution directly, either as a boundary-value or initial-value problem, all the equations in reactor theory that we will encounter all can be derived from the transport equation in one approximation or another.

Neutron Current

The term current has been used to denote a stream of particles flowing in a certain direction; it frequently appears in discussions of scattering of a particle beam or some kind of transport process. To be precise, one should begin with the definition in terms of the neutron flux ϕ which, as we have just seen, is the solution to the neutron transport equation. Since the current and the flux have the same dimensions, what then is the difference between these two quantities? Let

$$\begin{aligned} \underline{J}(\underline{r}, E, t) &\equiv \int_{\underline{\Omega}} \underline{\Omega} \phi(\underline{r}, E, \underline{\Omega}, t) d\Omega \\ &= \int_{\underline{\Omega}} v n(\underline{r}, E, \underline{\Omega}, t) d\Omega \end{aligned} \quad (8.6)$$

Recall the meaning of a particle flux as

$$v n(\underline{r}, E, \underline{\Omega}, t) dE d\Omega \Delta t dA \cos \theta = \text{expected no. in } dE \text{ about } E \text{ and } d\Omega \\ \text{about } \underline{\Omega} \text{ crossing } dA \text{ during } \Delta t \quad (8.7)$$

Let $J_+(\underline{r}, E, t) dE dA \Delta t = \text{expected no. in } dE \text{ about } E \text{ crossing } dA \text{ during } \Delta t \\ \text{from '-' to '+' in the sense of a normal vector } \underline{\hat{n}}$

$$= \int_{\underline{\hat{n}} \cdot \underline{\Omega} \geq 0} \underline{\hat{n}} \cdot \underline{\Omega} \phi(\underline{r}, E, \underline{\Omega}, t) dE d\Omega \Delta t dA \quad (8.8)$$

or,

$$J_+(\underline{r}, E, t) = \int_{\hat{n} \cdot \underline{\Omega} \geq 0} \hat{n} \cdot \underline{\Omega} \phi(\underline{r}, E, \underline{\Omega}, t) d\Omega \quad (8.9)$$

Similarly, those going in the opposite direction become

$$J_-(\underline{r}, E, t) = \int_{\hat{n} \cdot \underline{\Omega} \leq 0} (-\hat{n} \cdot \underline{\Omega}) \phi(\underline{r}, E, \underline{\Omega}, t) d\Omega \quad (8.10)$$

We can define the vector $\underline{J}(\underline{r}, E, t)$ such that

$$\hat{n} \cdot \underline{J}(\underline{r}, E, t) = J_+(\underline{r}, E, t) - J_-(\underline{r}, E, t) = \hat{n} \cdot \int_{\underline{\Omega}} \underline{\Omega} \phi(\underline{r}, E, \underline{\Omega}, t) d\Omega \quad (8.11)$$

which is consistent with (8.6), and allows us to arrive at the interpretation

$$\hat{n} \cdot \underline{J}(\underline{r}, E, t) dE dA \Delta t = \text{net no. neutrons in } dE \text{ about } E \text{ crossing } dA \text{ from '-' to '+' during } \Delta t \quad (8.12)$$

The difference between $\hat{n} \cdot \underline{J}$ and J_+ lies in the word net.

Problem Classification

The neutron distribution is a function of 7 variables, 3 for the neutron position, 3 for the velocity, or one for energy and two for direction of travel, plus time. One can define several reduced distributions by integrating out one or more of the variables, or by setting a variable such as the energy at a particular value. By doing this one breaks down the transport problem into several simpler problems such as neutron slowing down, neutron diffusion, neutron thermalization, and one-speed neutron transport. Each then can be studied separately with further simplifying approximations. Not only does each problem have physical interest by itself, but also the problems can be combined to give understanding of the essence of neutron transport.

For steady-state problems one simply postulates that the neutron distribution and the external source are both time-independent. The left-hand side of (8.5) vanishes; what is left is the time-independent neutron transport equation. Two kinds of problems can be investigated, one with an external source and the other without. The former corresponds to a subcritical reactor system since a source is needed to maintain a steady-state neutron density. The latter describes a critical reactor, a system in which one can have steady-state neutron density in the absence of a source.

Suppose one is not interested in the neutron direction of travel, then (8.5) can be integrated over $\underline{\Omega}$. In steady-state and in the absence of a source, this gives

$$-\underline{\nabla} \cdot \underline{J}(\underline{r}, E) + \int dE' \Sigma_s(E') \phi(\underline{r}, E') F(E' \rightarrow E) + \nu f(E) \int dE \Sigma_f(E') \phi(\underline{r}, E') - \Sigma_t(E) \phi(\underline{r}, E) = 0 \quad (8.13)$$

where we have used (8.6). This equation describes the distribution of neutrons in configuration space and energy. It is not a closed equation in that it involves two unknown quantities, the neutron flux and the net current. To take another step in simplification, one commonly makes the approximation that the current is related to the flux by the Fick's law of diffusion,

$$\underline{J} = -D \underline{\nabla} \phi \quad (8.14)$$

which then leads to the diffusion equation for a multiplying system in steady state,

$$[D \nabla^2 - \Sigma_t] \phi(\underline{r}, E) + \nu f(E) \int dE' \Sigma_f(E') \phi(\underline{r}, E') + \int dE' \Sigma_s(E') \phi(\underline{r}, E') F(E' \rightarrow E) = 0 \quad (8.15)$$

This is the energy-dependent diffusion equation, still not the simplest diffusion equation one can solve. Since this equation also treats the process of neutron energy moderation (neutron slowing down), further reductions can be considered.

At this stage one can go in two directions of simplification. One is to get rid of the energy dependence by integrating over E. The result is then the problem of neutron diffusion

$$\bar{D} \nabla^2 \bar{\phi}(\underline{r}) + [\nu \bar{\Sigma}_f - \bar{\Sigma}_a] \bar{\phi}(\underline{r}) = 0 \quad (8.16)$$

where we are using the overhead bar denotes two kinds of energy-integrated quantities, an effective flux averaged cross section and the integrated flux

$$\bar{\Sigma} = \frac{\int dE \Sigma(E) \phi(\underline{r}, E)}{\int dE \phi(\underline{r}, E)} \quad \text{and} \quad \bar{\phi}(\underline{r}) = \int \phi(\underline{r}, E) dE \quad (8.17)$$

This is the starting point for the discussion of neutron diffusion. For simplicity we will not bother to write out explicitly the overhead bar, but the origin of (8.16) should be kept in mind. We will return to consider the different solutions to this equation, including the treatment of boundary conditions. Notice that (8.16) is still a balance relation - the difference between what flows into and out of a volume element (by diffusion) is balanced by the difference between production and absorption in the volume element.

The other simplification of (8.15) is to get rid of the spatial dependence by integrating over the volume of the system. The first term in (8.15) then vanishes because the integral gives

$$\int_V D \nabla^2 \phi(\underline{r}, E) d^3 r = -D \int_V \nabla \cdot \underline{J}(\underline{r}, E) d^3 r = -D \int_S \underline{J}(\underline{r}, E) \cdot d\underline{S}$$

which is proportional to the current integrated over the surface of the system. The latter is taken to be zero for large (infinite) systems. The remaining terms in (8.15) become

$$v f(E) \int \Sigma_f(E') \phi(E') dE' - \Sigma_t(E) \phi(E) + \int \Sigma_s(E') \phi(E') F(E' \rightarrow E) dE' = 0 \quad (8.18)$$

where we continue to use the same symbol for the spatially integrated flux,

$$\phi(E) = \int_V \phi(\underline{r}, E) d^3 r \quad (8.19)$$

Eq.(8.18) is the starting point for the discussion of neutron slowing down. This, too, is a balance in energy space - the loss of neutrons at energy E by all kinds of interactions, represented by Σ_t , is balanced by gains from neutrons produced directly from fission and neutrons which scatter into energy E from some other energy. We will discuss the various solutions to (8.18) in the next lecture.

In addition to (9.16) and (9.18) two other equations can be derived from the transport equation, (9.5). Suppose we assume that all neutrons can have only energy, E_0 . This is tantamount to setting the following,

$$\phi(\underline{r}, E, \underline{\Omega}) = \phi(\underline{r}, \underline{\Omega}) \delta(E - E_0) \quad (8.20)$$

$$F(E' \underline{\Omega}' \rightarrow E, \underline{\Omega}) = F(\underline{\Omega}' \rightarrow \underline{\Omega}) \delta(E - E_0) \quad (8.21)$$

$$f(E) = \delta(E - E_0) \quad (8.22)$$

$$\Sigma(E) = \Sigma, \text{ a constant} \quad (8.23)$$

Here $\delta(x)$ is the Dirac delta function - it is zero everywhere except at the point where its argument vanishes, and there its value is infinite (the integral of the delta function is unity provided the point where it is infinite lies within the range of integration). Some properties of $\delta(x)$ are:

$$\int_{a-\varepsilon}^{a+\varepsilon} \delta(x-a) dx = 1, \quad \int_{a-\varepsilon}^{a+\varepsilon} f(x) \delta(x-a) dx = f(a) \quad (8.24)$$

$$\delta(x) = \delta(-x), \quad x\delta(x) = 0, \quad \delta(ax) = \frac{1}{a}\delta(x) \quad (8.25)$$

Inserting (8.20) - (8.23) into the transport equation (8.5) and integrating over E, we get

$$[\Sigma_t + \underline{\Omega} \cdot \underline{\nabla}] \phi(r, \underline{\Omega}) = \frac{v\Sigma_f}{4\pi} \int_{\underline{\Omega}'} d\underline{\Omega}' \phi(r, \underline{\Omega}') + S(r, \underline{\Omega}) + \Sigma_s \int_{\underline{\Omega}'} d\underline{\Omega}' \phi(r, \underline{\Omega}') F(\underline{\Omega}' \rightarrow \underline{\Omega}) \quad (8.26)$$

This is the one-speed, time-independent transport equation. It can be simplified further if we restrict our attention to a slab system (system is infinite in two directions and finite in the third, or x, direction). Let the angle between $\underline{\Omega}$ and the x-axis be the polar angle θ . Then,

$$\underline{\Omega} \cdot \underline{\nabla} \phi = \mu \frac{\partial \phi}{\partial x} + \sin \theta \cos \varphi \frac{\partial \phi}{\partial y} + \sin \theta \sin \varphi \frac{\partial \phi}{\partial z} \quad (8.27)$$

We then integrate over y and z, and the azimuthal angle φ to obtain

$$[\mu \frac{\partial}{\partial x} + \Sigma_t] \phi(x, \mu) = \frac{v\Sigma_f}{2} \int_{-1}^1 d\mu' \phi(x, \mu') + \Sigma_s \int_{-1}^1 d\mu' \phi(x, \mu') F(\mu_o) + S(x, \mu) \quad (8.28)$$

where

$$\phi(x, \mu) = \iint dydz \int_0^{2\pi} d\varphi \phi(xyz\mu\varphi) \quad (8.29)$$

and $\mu = \cos \theta$, and we have put $F(\underline{\Omega}' \rightarrow \underline{\Omega}) = F(\underline{\Omega} \cdot \underline{\Omega}') \equiv F(\mu_o) / 2\pi$. Eq.(8.28) is the one-speed transport equation for a slab. It is the simplest transport equation which one can consider. It is however still as rigorous as the original transport equation since we have made no approximations in the reduction. Eq.(8.28) can be solved in an infinite-medium problem or in problems involving boundaries. These solutions are useful mainly for checking approximate solutions and also in cases where energy dependence is of no interest.

The P₁ Approximation in Transport Theory

Eq. (8.28) is still an integro-differential equation that cannot be solved easily. One of the most common methods of extracting a more tractable description of transport is to expand the distribution function in a series of angular functions and then truncate the series. This is reminiscent of the use of partial wave expansion as a means of reducing the Schroedinger equation to a more manageable form. For this discussion we can ignore the fission without any loss of generality. We write

$$\phi(x, \mu) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \phi_{\ell}(x) P_{\ell}(\mu) \quad (8.30)$$

$$F(\mu_o) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} F_{\ell} P_{\ell}(\mu_o) \quad (8.31)$$

where $P_{\ell}(\mu)$ is the ℓ th order Legendre polynomial. This expansion leads to an infinite set of coupled equations for the functions

$$\phi_{\ell}(x) = \int_{-1}^1 d\mu P_{\ell}(\mu) \phi(x, \mu) \quad (8.32)$$

$$F_{\ell} = \int_{-1}^1 d\mu P_{\ell}(\mu) F(\mu) \quad (8.33)$$

The first two such equations are

$$\frac{d\phi_1(x)}{dx} + \Sigma_t \phi_o(x) = \Sigma_s F_o \phi_o(x) + S_o(x) \quad (8.34)$$

$$\frac{2}{3} \frac{d\phi_2(x)}{dx} + \frac{1}{3} \frac{d\phi_o(x)}{dx} + \Sigma_t \phi_1(x) = \Sigma_s F_1 \phi_1(x) \quad (8.35)$$

If somehow we are justified in ignoring the term containing $\phi_2(x)$, then we would have a closed set of equations in which we can solve for the flux coefficients, ϕ_o and ϕ_1 ,

$$\frac{d\phi_1(x)}{dx} + (\Sigma_t - \Sigma_s F_o) \phi_o(x) = S_o(x) \quad (8.36)$$

$$\frac{1}{3} \frac{d\phi_o(x)}{dx} + (\Sigma_t - \Sigma_s F_1) \phi_1(x) = 0 \quad (8.37)$$

The truncation of the Legendre polynomial expansion at $\ell=1$ is called the P_1 approximation. In general, retaining the $N+1$ terms in the series leads to a coupled set of $N+1$ equations – the P_N approximation.

The two scattering kernel coefficients which appear in the P_1 equations have quite simple meanings,

$$F_o = \int_{-1}^1 d\mu F(\mu) = 1 \quad (\text{particle conservation}) \quad (8.38)$$

$$F_1 = \int_{-1}^1 d\mu \mu F(\mu) = \langle \mu \rangle \quad (8.39)$$

An immediate consequence of the P_1 approximation is that we obtain a relation between the current and the gradient of the flux from (8.37),

$$\phi_1(x) = -\frac{1}{3}[\Sigma_t - \Sigma_s \langle \mu \rangle]^{-1} \frac{d\phi_0(x)}{dx} \quad (8.40)$$

where ϕ_0 and ϕ_1 are the neutron flux distribution in position space (position flux) and the neutron net current, respectively. Such a relation defines a transport coefficient, in this case, the neutron diffusion coefficient,

$$D = [3(\Sigma_t - \Sigma_s \langle \mu \rangle)]^{-1} \quad (8.41)$$

Eq.(9.40) is the P_1 approximation to the Fick's law of diffusion, generally written as

$$\underline{J} = -D\underline{\nabla}\phi \quad (8.42)$$

This relation is also well known in the kinetic theory of gases, with one difference. In neutron transport, the diffusion coefficient has dimension of cm, whereas in kinetic theory its dimension would be cm^2/sec . This distinction arises from a factor of the thermal speed, that is, in kinetic theory one uses number density rather than flux.

If absorption is weak, $\Sigma_a / \Sigma_t \ll 1$, the diffusion coefficient simplifies to

$$D \approx \frac{1}{3\Sigma_s(1 - \langle \mu \rangle)} \equiv \frac{1}{3\Sigma_{tr}} \quad (8.43)$$

with Σ_{tr} being an effective 'transport cross section'. Since $\langle \mu \rangle$ is average of the cosine of the scattering angle in LCS, it would have a larger value if the scattering were preferentially biased in the forward direction. Then D and the net current would increase as one would intuitively expect. On the hand, if the medium has a large scattering cross section, D would decrease since the neutron would be scattered more frequently thus impeding its forward progress. These simple physical interpretations apply to neutron diffusional transport as well as particle diffusion in general. Another remark is that we can go back to the transport equation without making the P_1 approximation, and derive a Fick's law relation between the current and the flux gradient. This however would take us beyond the scope of the present discussion.