

22.54 Neutron Interactions and Applications (Spring 2005)
Lecture 9 (3/10/05)

Neutron Slowing Down

References --

J. R. Lamarsh, *Introduction to Nuclear Reactor Theory* (Addison-Wesley, Reading, 1966)

We now turn our attention to the topic of neutron energy moderation, the slowing down of a fast neutron to the thermal energy region. The equation that describes the slowing down process, as we can expect by now, can be obtained by reducing the transport equation, Eq. (8.5) in Lec8. There are two simple ways to get rid of the spatial dependence of the flux. One is to integrate the transport equation over a large system and then take the system size to be infinite,

$$\phi(E) = \int \phi(\underline{r}, E) d^3r \quad (9.1)$$

$$\int \underline{\nabla} \cdot \underline{J} d^3r = \int_S (\hat{n} \cdot \underline{J}) dS \rightarrow 0 \quad (9.2)$$

The other is to consider a "point reactor",

$$\phi(\underline{r}, E) = \phi(E) \delta(\underline{r}) \quad (9.3)$$

and again integrate over space (along with invoking Fick's rule),

$$\int_{-\infty}^{\infty} dx \frac{\partial^2}{\partial x^2} \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} dk e^{ikx} = - \int_{-\infty}^{\infty} dk k^2 \delta(k) = 0 \quad (9.4)$$

In either case, one obtains the equivalent of Eq.(8.18),

$$\Sigma_t(E)\phi(E) = S(E) + \int dE' \Sigma_s(E')\phi(E')F(E' \rightarrow E) \quad (9.5)$$

which is known as the slowing down equation (for an infinite medium). In writing (9.5) we have combined the external source with the fission source in S(E). Without specifying the scattering kernel F, an equation having the same form as (9.5) is also applicable to the thermal energy region, where up-scattering of the neutron must be considered. In that case the energy balance problem would be called neutron thermalization (this will be the topic of a later lecture).

We will study (9.5) using the energy transfer kernel we have previously derived under the conditions of elastic scattering, target nucleus at rest, and isotropic scattering in CMCS (cf. Lec2-4). This kernel is perfectly valid in the slowing down region where the neutron loses energy every time it is scattered by the nucleus. Recall then

$$F(E' \rightarrow E) = \frac{1}{E'(1-\alpha)} \quad \alpha E' \leq E \leq E'$$

$$= 0 \quad \text{otherwise} \quad (9.6)$$

where $\alpha = (A-1)^2 / (A+1)^2$, and $A = M/m$.

Before discussing the solution of (9.5), we first consider the problem of estimating the escape probability $P(E' \rightarrow E)$ that a neutron, scattered at E' , will cross E , with no restriction on how large is the interval from E' to E and how many collisions are involved. Notice that $P(E' \rightarrow E)$ differs from $F(E' \rightarrow E)$ in that the latter is for one collision and is a distribution function in the variable E . (In contrast P is not a distribution function, although it is a probability.) The exercise of estimating $P(E' \rightarrow E)$ will also help to make clear this distinction. Since the energy range (E', E) is arbitrary, we divide this interval into a number of subintervals, $\Delta E_j, j = 1, \dots$, such that for each subinterval we can write

$$\{ p_a(\Delta E_j) \} = \{ \text{avg. no. coll. req'd to cross } \Delta E_j \} \times \{ \text{prob absorp per coll} \} \quad (9.7)$$

where $p_a(\Delta E_j)$ is probability of neutron absorption while crossing ΔE_j . The second term in (9.7), the probability of absorption each time the neutron collides, is just

$$\{ \text{prob absorp per coll} \} = \Sigma_a(\Delta E_j) = \Sigma_t(\Delta E_j) \quad (9.8)$$

Eq.(9.7) reveals the strategy we are following in making our estimate of P . To find the probability of no absorption while crossing the interval (E', E) , we write it as a product of probabilities,

$$P(E' \rightarrow E) = \prod_j [1 - p_a(\Delta E_j)] \quad (9.9)$$

where $p_a(\Delta E_j)$ is to be obtained according to (9.7) and (9.8). The advantage of doing this is that once we have (9.9), we can rewrite it as

$$\ln P = - \sum_j p_a(\Delta E_j) \rightarrow \int_E^{E'} p_a(E'') dE'' \quad (9.10)$$

using the properties, $\ln(ab) = \ln(a) + \ln(b)$, and $\ln(1-x) \sim -x$ for $x \ll 1$. Now we see the condition on the subinterval division, ΔE_j has to be chosen small enough to make $p_a(\Delta E_j)$ small compared to unity. Eq.(10.10) means the probability we are seeking is given by the expression,

$$P(E' \rightarrow E) = \exp\left(-\int_E^{E'} p_a(E'') dE''\right) \quad (9.11)$$

To turn this into an explicit expression we can use (9.7) and (9.8) to find $p_a(\Delta E_j)$, which means we need to make an estimate of the average number of collisions a neutron should make in going from energy E' to energy E .

A simple way to estimate the number of collisions a neutron should make, on average, in crossing an energy interval, we need only the average energy that a neutron will lose in a collision at a given energy. Suppose the neutron collision occurs at energy E' and the neutron has energy E after the collision. The probability of such an event is just given by the energy transfer kernel $F(E' \rightarrow E)$. The energy the neutron would have, on average, after a collision at E' then becomes

$$\bar{E} = \int EF(E' \rightarrow E) dE = (1+\alpha)E'/2 \quad (9.12)$$

where we have used (9.6). The energy loss per collision is $E' - \bar{E} = (1-\alpha)E'/2$. Thus the average number of collisions required to cross an interval ΔE is

$$\{\text{avg. no. coll. req'd to cross } \Delta E_j\} = \frac{2}{1-\alpha} \frac{\Delta E}{E} \quad (9.13)$$

With this result we have an explicit expression for the escape probability $P(E' \rightarrow E)$ from (9.11),

$$P(E' \rightarrow E) = \exp\left(-\frac{2}{1-\alpha} \int_E^{E'} \frac{\Sigma_a(E'')}{\Sigma_t(E'')} \frac{dE''}{E''}\right) \quad (9.14)$$

This expression contains the parameter α whose numerical value depends on mass number of the target nucleus A . In the case of a hydrogenous medium, $A = 1$ and $\alpha = 0$. For a heavy target nucleus, $A \gg 1$, then $\alpha \sim 1 - (4/A)$, and the coefficient in front of the integral in (10.14) becomes $A/2$.

Before closing this discussion, we take the opportunity to introduce another energy-like quantity, called the lethargy in the reactor physics literature,

$$u \equiv \ln(E_o / E) \quad (9.15)$$

where E_o is some constant energy. Notice that there is a one-to-one correspondence between energy and lethargy, the two variables move in opposite directions in that when the energy decreases the lethargy would increase. Given that we have just calculated the average energy loss per collision, it is perhaps not surprising that the corresponding average lethargy increase per collision turns out to be a rather useful quantity,

$$\langle \Delta u \rangle = \langle u - u' \rangle = \langle \ln(E'/E) \rangle = \int_{\alpha E'}^{E'} \ln(E'/E) F(E' \rightarrow E) dE = 1 + \frac{\alpha \ln \alpha}{1 - \alpha} \equiv \xi \quad (9.16)$$

In the two limits which are always of interest,

$$A = 1, \quad \alpha = 0, \quad \xi = 1$$

$$A \gg 1, \quad \alpha \sim 1, \quad \xi = 2/(A + 2/3) \sim 2/A$$

We can use (9.16) to estimate the average number of collisions to cross an interval ΔE ,

{no. coll. to cross ΔE } =

$$\frac{\ln(E) - \ln(E - \Delta E)}{\xi} = \frac{1}{\xi} \ln\left(\frac{E}{E - \Delta E}\right) = \frac{1}{\xi} \ln\left(1 + \frac{\Delta E}{E}\right) \sim \frac{1}{\xi} \frac{\Delta E}{E} \quad (9.17)$$

Notice that if we inset the estimate (9.17) into (9.7), we would have arrived at the estimate for P,

$$P(E' \rightarrow E) = \exp\left(-\int_E^{E'} \frac{\sum_a(E'') dE''}{\sum_t(E'') \xi E''}\right) \quad (9.18)$$

which agrees with (9.14) in the case of heavy target nucleus, but not when $A = 1$. We conclude that when $A > 1$ one can use either (9.14) or (9.18), whereas for $A = 1$ our present discussion indicates that (9.14) should be preferred since it involves fewer approximations.

We now turn our attention to the solution of the slowing down equation, Eq. (9.5), for the special case of hydrogenous medium, $A = 1$, a monoenergetic source at E_o , $S(E) = S_o \delta(E - E_o)$, and furthermore we neglect absorption. Eq.(9.5) then reads

$$\Sigma_s(E)\phi(E) = \frac{S_o}{E_o} + \int_E^{E_o} \Sigma_s(E')\phi(E') \frac{dE'}{E'} \quad (9.19)$$

This is a simple integral equation which can be solved readily. Let $G(E) = \Sigma_s(E)\phi(E)$, (9.5) is then of the form,

$$G(E) = const + \int_E^{E_o} G(E') \frac{dE'}{E'} \quad (9.20)$$

Differentiating both sides with respect to E gives,

$$\frac{dG(E)}{dE} = -\frac{G(E)}{E}, \text{ or } \frac{dG}{G} = -\frac{dE}{E} \quad (9.21)$$

So,

$$G(E) = c/E \quad (9.22)$$

where c is an integration constant. To find c, take the limit as E approaches E_o in which case G(E) should approach S_o , thus $c = S_o$.

We have shown that in the energy region of neutron slowing down, the flux is given by

$$\phi(E) = \frac{S_o}{\Sigma_s(E)E} \quad (9.23)$$

The "1/E" behavior of the flux is very characteristic of the neutron distribution during energy moderation by elastic collisions with the target nuclei. Eq.(9.23) has been obtained for the case of the hydrogenous medium. We can extend it to an arbitrary medium, $A > 1$, by writing in the same spirit as (9.17) and (9.18),

$$\phi(E) = \frac{1}{\Sigma_s(E)\xi E} \quad (9.24)$$

In summary, we have obtained an expression for the probability that a neutron will undergo energy moderation by elastic scattering across an arbitrary interval in the form of Eq. (9.18), which involves the quantity ξ , the average lethargy increase per collision. We have also shown that the flux in the slowing down region, the region of energy below the fission spectrum and above thermal energy where neutrons can be up-scattered, is 1/E.