22.54 Neutron Interactions and Applications (Spring 2003)
Lecture 3 (2/18/03)

Kinematics of Nuclear Reactions and the Q-Equation

References --
W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), Sec. 3.3.
R. D. Evans, *The Atomic Nucleus* (McGraw-Hill, New York, 1955), Chap 12.

When two particles collide, the entire event is described by the velocity vectors of the particles, two before and two after the collision. There are therefore 12 velocity components which can be regarded as 12 degrees of freedom of the problem. The number of independent degrees of freedom is of course less than 12; since momentum and energy must be conserved, this means there can be atmost 8 independent dgrees of freedom. If one further specifies the incoming particle velocity and take the target nucleus to be at rest, leaving only 2 degrees of freedom to be determined. We will see that the analysis of two-body collision amounts to a study of the relationship between these two degrees of freedom.

The purpose of this lecture is to discuss the *kinematics* of two-body collision. By this we mean we are interested in the energy-momentum or energy-angle relation involving only momentum and energy conservations, while saying nothing about the *dynamics* of the collision which is governed by the interaction potential. We consider an incoming particle (labeled 1) striking a target nucleus (2), causing a reaction in which an outgoing particle (3) is emitted at angle θ relative to the incoming direction, and the product nucleus (4) recoils. The situation is depicted in Fig. 3-1. To keep matters as simple as possible, we will assume that the target nucleus is initially at rest. While this simplifies considerably the analysis, we should keep in mind that the assumption is justified only when energy of the incoming particle is large compared to the $k_B T$, where T is the target temperature. Given that we are interested in applying our results to neutron scattering, the assumption is valid for neutrons during their slowing down from energies above thermal, but not for neutrons at thermal energies.

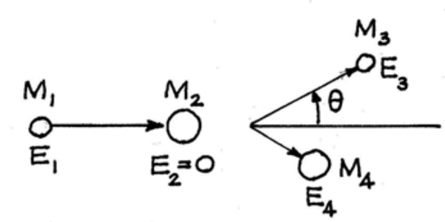

Fig. 3.1. Schematic of two-body collision with target nucleus at rest. Outgoing particle is emitted at an angle θ relative to the direction of incidence. Product nucleus recoils.

We will now set up an equation which incorporates the conservation of energy and momentum in a way that relates the outgoing energy E_3 to the angle of emission θ. Energy conservation means

$$(E_1 + M_1c^2) + M_2c^2 = E_3 + M_3c^2 + E_4 + M_4c^2 \tag{3.1}$$

From momentum conservation we have

$$p_4^2 = (\underline{p}_1 - \underline{p}_3)^2$$

$$= \; p_1^2 + p_3^2 - 2p_1 p_3 \cos\theta \qquad (3.2)$$

Recalling the definition of Q-value (2.1 and 2.2),

$$Q = (M_1 + M_2 - M_3 - M_4)c^2$$

$$= \; E_3 + E_4 - E_1 \qquad (3.3)$$

we combine (3.1) - (3.3) to obtain

$$Q = E_3(1 + M_3 / M_4) - E_1(1 - M_1 / M_4) - (2 / M_4)\sqrt{M_1 M_3 E_1 E_3}\cos\theta \qquad (3.4)$$

which is known as the *Q-equation*. Notice that this is not really an equation to be used to calculate Q, rather it is an equation which relates E_3 and θ, with Q being merely a constant whose value is known when reaction is specified. In (3.4) the energies E_i and θ are in LCS while Q is independent of coordinate system. Often this equation is used when E_1, the masses, and the Q-value are known, then one is interested in solving this equation for E_3 in terms of θ, or vice versa.

Seeing that (3.4) contains a term with $(E_3)^{1/2}$ we can regard it as a quadratic equation in $(E_3)^{1/2}$, which in general has two solutions. Since E_3 is an energy, the physically acceptable solution must be real and positive. An interesting situation arises when both solutions satisfy this condition [see, Evans, pp. 413-415 and Meyerhof, p. 178]. We will return later to discuss this rather special case. For now we consider the implications of (5.4) for neutron inelastic and elastic scattering.

Neutron Inelastic Scattering

We set $M_1 = M_3 = m$, the neutron mass, and $M_2 = M$, the target mass. As we have seen in Lec 2, this is an endothermic reaction, with $Q = -E^*$, the excitation energy imparted to the target nucleus. The product nucleus then has mass $M_4 = M^*$, with $M^* = M + E^*/c^2$. If we take $\theta = 0$ and $E_3 \sim 0$, we find

$$E_1 = E^*\left(\frac{M_4}{M_4 - M_1}\right) \simeq E^*\left(\frac{M + m}{M}\right) \qquad (3.5)$$

This is a good approximation to the *threshold energy*, the minimum possible value of E_1 for which the reaction can take place. It is expected that the threshold energy should be slightly greater than the excitation energy. This is because some of the incoming energy has to go into the energy of the center-of-mass which is not available for reaction.

Neutron Elastic Scattering

In this case Q = 0, and (3.4) becomes

$$E_3 - \frac{2m}{M + m}\sqrt{E_1 E_3}\cos\theta - \frac{M - m}{M + m}E_1 = 0 \qquad (3.6)$$

with two general solutions,

$$\sqrt{E_3} = \frac{m}{M + m}\cos\theta\sqrt{E_1} \pm \frac{1}{M + m}[m^2\cos^2\theta + M^2 - m^2]^{1/2}\sqrt{E_1}$$

or, $$\sqrt{E_3} = \frac{m}{M+m}\left(\cos\theta + [(M/m)^2 - \sin^2\theta]^{1/2}\right)\sqrt{E_1} \tag{3.7}$$

where we have chosen the upper sign for the physical solution. In the case of forward scattering (equivalent to no scattering), θ =0, then $E_3 = E_1$, as would be expected. For backward scattering, $\theta = \pi$, we find

$$E_3 = \alpha E_1, \qquad \text{where } \alpha = \frac{(M-m)^2}{(M+m)^2} \tag{3.8}$$

These two cases represent the two extremes. A neutron undergoing elastic scattering at energy E therefore has its outgoing energy in the range $(E, \alpha E)$.

It is very useful to have a simple relation between E_3 and the scattering angle θ. Eq.(3.7) is already such a relation; however, it is somewhat complicated algebraically. A simpler relation exists if we are willing to consider the scattering angle in center-of-mass coordinate system.

Elastic Scattering in CMCS

The analysis of two-body collision in the center-of-mass coordinate system offers significant simplification, not only in the kinematics of the collision but also in the dynamics when spherically symmetric or central interaction potentials are involved. We will now demonstrate the former by obtaining a simpler energy-angle relation than (3.7). The latter advantage will be apparent when we discuss the partial-wave method for calculating the cross section (see Lec 4).

Fig. 3.2 shows a typical collision between an incoming neutron and a stationary target nucleus (subscript A) depicted in the laboratory coordinate system (LCS) and in the center-of-mass coordinate system (CMCS).

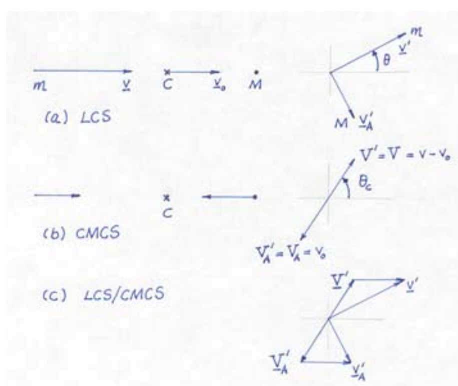

Fig. 3.2. Velocities vectors before and after a two-body elastic collision in LCS (a) and CMCS (b). Composite (c) shows the relation between LCS and CMCS velocities. Also shown are the scattering angles in the two coordinate systems.

We adopt the notation of labeling the velocities in LCS and CMCS in lower and upper cases respectively, and denote the velocities after the collision as primed. Also, since incoming and outgoing particles are the same neutron, no subscript is used for the velocities for simplicity. For the target and product nucleus, also the same particle, a subscript A is used. As seen in Fig. 3.2, in CMCS the center-of-mass is at rest while the incoming particle and the target nucleus move toward each other with speeds $V = v - v_o$ and v_o, respectively. The effect of the elastic collision is simply to rotate the two pre-collision velocities in CMCS, $\underline{V}$ and $\underline{V}_A$, such that after the collision they move off into opposite directions (to conserve momentum).

Because the target nucleus is at rest, the velocity of the center-of-mass, $\underline{v}_o$, is just $(m/M+m)\underline{v}$. The velocity of a particle in CMCS is defined to be $\underline{V} = \underline{v} - \underline{v}_o$ for pre- as well post-collision, the latter being indicated by a prime. Since the magnitudes of $\underline{V}$ and $\underline{V}_A$ do not change during the collision, one can readily determine the post-collision velocities in LCS by construction as indicated in Fig. 3.2(c). At the same time, one can see the relation between the two scattering angle, θ and its CMCS counterpart θ_c.

In view of (3.7) we read off from Fig. 3.2(c) the relation between the post-collision velocities of the scattered neutron in LCS and CMCS. In terms of their magnitudes the relation is

$$v'^2 = (\underline{V}' + \underline{v}_o)^2$$

$$= V^2 + v_o^2 + 2Vv_o \cos\theta_c \tag{3.9}$$

or,

$$E' = \frac{1}{2}E\{(1+\alpha) + (1-\alpha)\cos\theta_c] \tag{3.10}$$

with $\alpha = (A-1)^2/(A+1)^2$, A = M/m. This is the simple energy-relation we are seeking; it is actually identical to the square of (3.7), when converted to the same notation in labeling the various speeds,

$$E_3 = \frac{1}{(A+1)^2}E_1\left(\cos^2\theta + A^2 - \sin^2\theta + 2\cos\theta[A^2 - \sin^2\theta]^{1/2}\right) \tag{3.11}$$

which contains the LCS scattering angle θ instead of θ_c. To demonstrate this equivalence one needs a relation between the two angles. This can be obtained from Fig. 3.2(c),

$$\cos\theta = (v_o + V_3\cos\theta_c)/v_3$$

$$= \frac{1 + A\cos\theta_c}{\sqrt{A^2 + 1 + 2A\cos\theta_c}} \tag{3.12}$$

Solutions to the Q-equation (3.4) for reactions of definite Q

We return to discuss further the properties of the general solutions to Q-equation, treated as a quadratic equation in $(E_3)^{1/2}$,

$$[\sqrt{E_3}]_\pm = s \pm \sqrt{s^2 + t} \tag{3.13}$$

with

$$s = \frac{\sqrt{M_1 M_3 E_1}}{M_3 + M_4}\cos\theta \tag{3.14}$$

$$t = \frac{M_4 Q + (M_4 - M_1)E_1}{M_3 + M_4} \tag{3.15}$$

Again, the condition for physical solutions is that $(E_3)^{1/2}$ must be *real and positive*. The consequences of this condition are interesting to examine. When the masses are fixed, Q is known. Then (3.13) becomes a relation between the energy of the outgoing particle and the angle of emission, since usually the energy of the incoming particle is also given. We will distinguish between exothermic and endothermic reactions. In the latter situations (3.13) becomes a condition on the critical value of E_1 for which the reaction is energetically possible.

<u>Exothermic reaction (Q > 0), or $E_3 + E_4 > E_1$</u>. An example would be $B^{10}(\alpha, p)C^{13}$, Q = 4 MeV. At very low incident energy E_1, we can set s ~ 0. Then, $E_3 = [M_4/(M_3 + M_4)]Q$ and correspondingly, $E_4 = [M_3/(M_3 + M_4)]Q$. One can show that in this case the outgoing particle and product nucleus go off in opposite directions, as one should expect from momentum conservation. As E_1 increases, since $M_1 < M_4$ (if we choose C^{13} to be the product nucleus), then t > 0 and only the upper sign in (3.13) is acceptable. But the assignment of outgoing particle is arbitrary. Suppose we choose C13 as the outgoing particle (3), then (3.15) gives $t = (Q - 3E_1)/14$. For $E_1 > Q/3$, t will be negative, which means we now have two physical solutions for $(E_3)^{1/2}$. We will return to see what is meaning of a double-valued solution. Another noteworthy consequence is that for $\cos\theta < 0$, s < 0, and with $E_1 > Q/3$, $(E_3)^{1/2}$ is always negative. This means that no C13 nucleus will be emitted at $\theta \geq \pi/2$.

<u>Endothermic reaction (Q < 0, threshold)</u>
The inverse of all exothermic reactions must be endothermic. From the foregoing example, we have $C^{13}(p,\alpha)B^{10}$, Q = -4 MeV. At low incident energy, s ~ 0, t < 0 so $(E_3)^{1/2}$ is imaginary. This indicates that there is no enough kinetic energy brought in by particle 1 to make up the required mass increase. We define the threshold energy as

$(E_1)_{thres} \equiv$ lowest incoming energy at which reaction can occur

As E_1 is increased, t will become less and less negative, while the most positive s can be for a given E_1 is when the scattering is in the forward direction $(\theta = 0)$ which is effectively no scattering. This means for increasing E_1 the reaction first becomes allowable when

$$s^2 + t = 0 \tag{3.16}$$

This condition then reflects back on a critical value for E_1, with θ arbitrary. Let us denote this special value as $(E_1)_\theta$. From (3.16) we obtain

$$(E_1)_\theta = -Q\left[\frac{M_3 + M_4}{M_3 + M_4 - M_1 - (M_1 M_3/M_4)\sin^2\theta}\right] \tag{3.17}$$

Eq. (3.17) shows that the smallest value of E_1 occurs at $\theta = 0$, for which the denominator is largest. The threshold value is therefore

$$(E_1)_{thres} = -Q\frac{M_3 + M_4}{M_3 + M_4 - M_1} \tag{3.18}$$

which is seen to be somewhat different from (3.5). To reconcile this difference we note that since $M_1 + M_2 = M_3 + M_4 + Q/c^2$, to a good approximation (3.18) simplifies to

$$(E_1)_{thres} \simeq -Q\frac{M_1 + M_2}{M_2} \tag{3.19}$$

which then agrees with the approximate expression given by (3.5). Both (3.5) and (3.19) point to the intuitively reasonable result that the threshold energy should be greater than the excitation energy or Q-value in a reaction. This is because a fraction of the incoming kinetic energy goes into moving the center-of-mass, which is therefore NOT available for driving the reaction. One needs an amount of kinetic energy that is larger than the Q value by approximately the ratio $(M_1+M_2)/M_2$. In other words, the amount in excess of Q that is needed is that given to the center-of-mass, or $(M_1/M_2)Q \sim (M_1/M_2)E_1$.

Double-valued solutions to the Q-equation

If E_1 and θ are such that the condition

$$0 \le s^2 + t \le s^2 \tag{3.20}$$

which is equivalent to

$$-s^2 \le t \le 0 \tag{3.21}$$

is satisfied, this is sufficient for the Q-equation to have double-valued solutions. Notice that the lower limit in (3.20) ensures that E_3 is real, while the upper limit ensures that it is positive. These two limits imply corresponding limits on the incoming kinetic energy E_1. From (3.21) the upper limit gives

$$E_1 \le -Q\frac{M_4}{M_4 - M_1} \tag{3.22}$$

while one obtains from the lower limit, after a bit of manipulation,

$$E_1 \ge -Q\left(\frac{M_3 + M_4}{M_3 + M_4 - M_1 - (M_1 M_3 / M_4)\sin^2\theta}\right) = (E_1)_\theta \tag{3.23}$$

At $\theta = \pi/2$,

$$(E_1)_{\pi/2} = -Q\frac{M_4(M_3 + M_4)}{M_3 M_4 + M_4^2 - M_1 M_3 - M_1 M_4}$$

$$= -Q\frac{M_4}{M_4 - M_1} \tag{3.24}$$

Thus the condition for double-valued solutions can be put into the form,

$$(E_1)_\theta \le E_1 \le (E_1)_{\pi/2} \tag{3.25}$$

One consequence of (3.25) is that in the domain of double-valued solutions no outgoing particle can be emitted at an angle θ greater than $\pi/2$. Having an angle of emission greater than $\pi/2$ makes $t > 0$, which in turn puts E_3 into the single-valued domain. Physically this means that the kinetic energy of the reaction products is large enough that M_3 can be projected into the backward direction in LCS. For this to occur, it is also necessary that $M_3 < M_2$. The 'heavier fragment' (product nucleus) in an endothermic reaction can never be projected in the backward direction. Fig. 3.3 summarizes our discussion of the different domains of physical solutions to the Q-equation in the case of endothermic reaction.

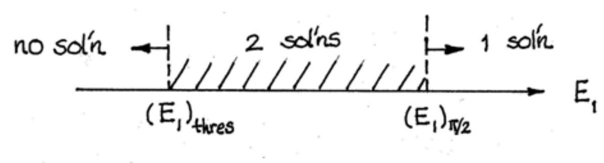

Fig. 3.3. Domain of double-valued solutions to the Q-equation is bounded by the domain of no solution from below at the threshold energy, Eq.(3.18), and by the domain of single-valued solutions at $(E_1)_{\pi/2}$, Eq.(3.24).

Another way to see the physical implication of the double-valued solutions is illustrated in Fig. 3.4. For an endothermic reaction, the meaning of these solutions is that a particle can emitted at a certain angle with two different energies. It is important to note that this occurs in LCS, which is an example for the violation of (3.10) - perhaps not surprising since (3.10) is for elastic scattering , but NOT in CMCS. As seen in Fig. 3.4, the two corresponding speeds in CMCS have the same magnitude, which means that the one-to-one correspondence between scattering angle and post-collision speed still holds in CMCS.

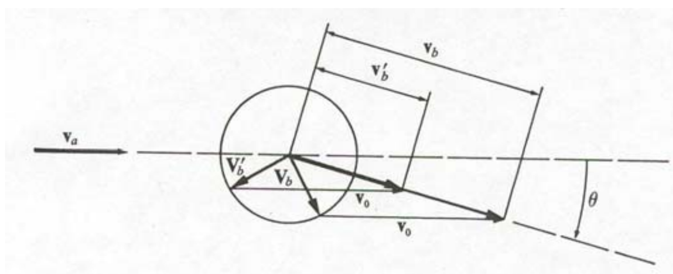

Fig. 3.4. Velocity diagram for an endothermic reaction in which the outgoing particle (b) appears at a certain angle θ with two distinct speeds, v_b and $v_b^{'}$, in LCS. In this diagram [from Meyerhof, p. 178] the notation differs from the system we have adopted in that all velocities are postcollision, and the prime simply denotes a different velocity.

We close this lecture with an example which illustrates further the relation among three principal quantities in the present discussion, the incoming and outgoing particle energies, and the angle of emission. The example is the reaction $Li^7(p,n)Be^7$, Q = -1.646 MeV [Evans, p. 416]. The threshold energy, in this case the incoming proton energy, is

$$(E_1)_{thres} = 1.646\frac{1+7}{7} = 1.881 \text{ MeV.}$$

The corresponding outgoing neutron energy is

$$E_3 = s^2 = \frac{1}{64}1.881 = 0.0294 \text{ MeV}$$

Interestingly this is not the lowest energy of the neutron that can be emitted, since the proton energy at which the neutron is barely emitted, $E_3 = 0$ (s=0, t=0), is

$$(E_1)_{\pi/2} = 1.646\left(\frac{8}{7-1/8}\right) = 1.915$$

In the range of proton energies between $(E_1)_{thres}$ and $(E_1)_{\pi/2}$, only certain emission angles are allowed; these lie in a cone which increases with increasing proton energy, as shown in Fig. 3.5 [Evans, p. 416].

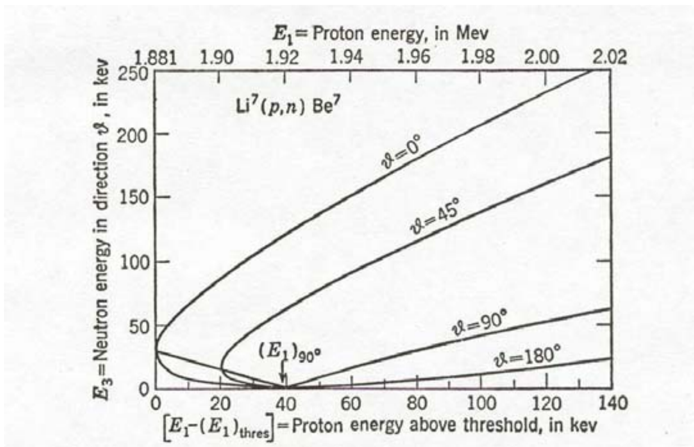

Fig. 3.5. Energetics of an endothermic reaction, $Li^7(p,n)Be^7$, Q = -1.646 MeV. All energies and angles are in LCS. Reaction first takes place at the threshold proton energy of 1.881 MeV, with neutron emitted at 0.0294 MeV. As proton energy is increased, neutron is emitted within a small forward cone, the extent of which increases with increasing proton energy. At proton energy of 1.920 MeV, all directions of neutron emission are allowed. One sees that doubled-valued solutions are allowed for emission at an angle less than 90^o, and in the range between 90^o and 180^o only single-valued solution is possible.

Notice that in this range, we have double-valued solutions - two neutron energies emitted for a given proton energy and emission angle. For proton energies greater than $(E_1)_{\pi/2}$, all angles are allowed, but now there is only one neutron energy for a given proton energy and emission angle.

The characteristic behavior of the curves shown in Fig. 3.5 is worth studying. The student can build up intuition by understanding the meaning of the various features, and asking yourself what would be the corresponding curves if the reaction were exothermic. The discussions in this lecture should allow you to deduce that by yourself.